

# Highest weight Harish-Chandra supermodules and their geometric realizations

## II. Representations of the supergroup

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### Abstract

This is the second part of a series of papers discussing the highest weight  $\mathfrak{k}_r$ -finite representations of the pair  $(\mathfrak{g}_r, \mathfrak{k}_r)$  consisting of  $\mathfrak{g}_r$ , a real form of a complex basic Lie superalgebra of classical type  $\mathfrak{g}$  ( $\mathfrak{g} \neq A(n, n)$ ), and the maximal compact subalgebra  $\mathfrak{k}_r$  of  $\mathfrak{g}_{r,0}$ . In this part we concretely realize these representations through superspaces of holomorphic vector bundles on a suitable Hermitian superspaces.

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# 1 Introduction

In [25] Harish-Chandra laid down the foundation for the theory of representation of real semisimple Lie algebras. He constructed modules, which are highest weight, both infinitesimally and globally. In [10] we initiated the study of the infinitesimal modules for the real forms of basic Lie superalgebras. The objective of the present work is to construct the global version of such modules. We realize the representation of a real form of a real supergroup, having as Lie superalgebra a real form of a basic Lie superalgebra, as the superspace of sections of a certain holomorphic vector bundle on the associated symmetric superspace, which is hermitian. Our construction is very explicit and in fact, in the end, we fully discuss the example of the Siegel superspace, which has an interest on its own.

This paper is organized as follows. In Sec. 2 we go through some preliminaries concerning supergeometry, which are not available elsewhere in the form we need. In particular, we discuss equivalent approaches to define vector bundles on supermanifolds, the real forms of complex Lie supergroups, some basic facts on the theory of covering spaces in supergeometry and finally some technical results on Fréchet superrepresentations.

Then in Sec. 3, we go to the heart of our treatment and construct infinite dimensional Harish-Chandra superrepresentations of a real form  $G_r$  of a simple complex Lie supergroup  $G$ . These representations have as their infinitesimal counterpart those that we have constructed in [10]. These representations are realized in the Fréchet superspace of sections of a (complex) line bundle on the quotient  $G/B^+$ , where  $B^+$  is a borel subsupergroup of  $G$ .

In Sec. 4, we discuss in detail the example of the Siegel superspace, which is interesting in itself for its possible applications to the theory of SUSY curves [47].

## 2 Supergeometry Preliminaries

In this section we introduce the basic definitions of supergeometry. We refer the reader for a complete basic treatment of this subject to [8] Ch. 4, 5, [43], [35]. One warning: we are interested both in the *analytic* and in the *differential category*; most of the treatments quoted above deal mainly with the differential setting. We shall make a note whenever the two categories differ in a substantial way.

### 2.1 Basic Definitions

Let the ground field be  $k = \mathbb{R}, \mathbb{C}$ .

A supermanifold (analytic or differentiable)  $M$  is understood as a pair  $(\widetilde{M}, \mathcal{O}_M)$  consisting of a topological space  $\widetilde{M}$  (denoted also as  $M_0$  or  $|M|$ , when the notation  $\widetilde{M}$  makes formulas too difficult to read), which is an ordinary manifold (real or complex analytic or differentiable), and a sheaf of superalgebras  $\mathcal{O}_M$  locally isomorphic to the tensor product of an exterior algebra and the ordinary sheaf of functions on  $\widetilde{M}$ .

There is an equivalent way to understand a supermanifold as a representable functor  $M$  from the category of supermanifolds to the category of sets:

$$(\text{smflds})^o \ni T \longmapsto M(T) = \text{Hom}(T, M)$$

with  $M(\phi)(f) = f \circ \phi$  on the arrows. In the smooth category we have the simple characterization that

$$\text{Hom}(T, M) = \text{Hom}(\mathcal{O}(M), \mathcal{O}(T))$$

$\mathcal{O}(M)$  denoting the superalgebra of global section on  $M$ . This fact is no longer true in the analytic category. The importance of the functor of points lies in Yoneda's Lemma, which establishes a one-to-one correspondence between the morphism of supermanifolds and the natural transformations of their functor of points.

We denote by  $(\text{smflds})_{\mathbb{C}}$  the category of complex analytic supermanifolds and by  $(\text{smflds})_{\mathbb{R}}$  the category of differentiable supermanifolds. Whenever the result or definition we are giving holds for both categories we shall just use  $(\text{smflds})$  to denote either of such categories (analytic or differentiable). We shall also use real analytic supermanifolds and we denote their category with  $(\text{smflds} - \text{anal})_{\mathbb{R}}$ .

For any supermanifold  $M$ , we can define  $M_{\text{red}}$  the *reduced manifold* as  $M_{\text{red}} = (\widetilde{M}, \mathcal{O}_M/J_M)$ , where  $J_M$  is the ideal sheaf in  $\mathcal{O}_M$  generated by the odd nilpotents.

If  $M = (\widetilde{M}, \mathcal{O}_M)$  is a supermanifold and  $U$  is open in  $\widetilde{M}$ , then  $(U, \mathcal{O}_M|_U)$  has a natural supermanifold structure. This prompts us to give the following definition.

**Definition 2.1.** Let  $N$  and  $M$  be supermanifolds. We say  $N$  is a *full open submanifold* of  $M$  if  $\widetilde{N}$  is open in  $\widetilde{M}$  and  $\mathcal{O}_N = \mathcal{O}_M|_{\widetilde{N}}$ .

We now state one of the fundamental results of the theory (see Theorem 4.2.5 in [8]).

**Theorem 2.2.** *Chart Theorem.* Let  $\widetilde{U}$  be open in  $k^{p|q}$ ,  $U = (\widetilde{U}, \mathcal{O}_{k^{p|q}}|_{\widetilde{U}})$  and  $M$  a supermanifold. There is a one to one correspondence between the morphisms  $M \rightarrow U$  and the set of  $p+q$ -uples of  $p$  even sections  $t_{i*}$  and  $q$  odd sections  $\theta_{j*}$  on  $M$ , such that  $(t_{1*}(x) \dots t_{p*}(x)) \in \widetilde{U}$  for all  $x \in \widetilde{M}$ .

**Example 2.3.** The functor of points of the supermanifold  $k^{m|n}$ . Let  $k^{m|n} = (k^m, \mathcal{O}_{k^{m|n}})$  denote the supermanifold whose reduced space is the topological space  $k^m$  and with supersheaf:

$$\mathcal{O}_{k^{m|n}}(U) = \mathcal{O}_{k^m}(U) \otimes \wedge(\xi_1, \dots, \xi_n), \quad U \subset k^m$$

Notice that in this example we are considering both the smooth and the holomorphic case at once. Its functor of points is given by:

$$(\text{smflds}) \rightarrow (\text{sets}), \quad k^{m|n}(T) = \text{Hom}(T, k^{m|n}).$$

By the Chart's Theorem (see 2.2), any morphism  $\psi: T \rightarrow k^{m|n}$  consists in the assignment, for each open subset  $U \subseteq k^m$ , of the images of coordinate sections  $x_i, \theta_j$  in  $\mathcal{O}_{k^{m|n}}(U)$ :

$$\begin{aligned}\psi_U^*: \mathcal{O}_{k^{m|n}}(U) &\longrightarrow \mathcal{O}_T(\tilde{\psi}^{-1}(U)) \\ x_i &\longmapsto f_i \\ \theta_j &\longmapsto \phi_j\end{aligned}$$

On the other hand,  $k^{m|n}$  admits global coordinates say  $y_i, \xi_j$ . Since,  $\psi^*(y_i|_U) = \psi^*(y_i)|_U$  we see that the assignment of  $m$  even and  $n$  odd global sections in  $\mathcal{O}(T)$  uniquely determines the pull-back. Hence  $k^{m|n}(T)$  can be identified with the set of  $m|n$ -uples  $(t_1, \dots, t_m, \theta_1, \dots, \theta_n)$ , where the  $t_i$ 's and  $\theta_j$ 's are, respectively, even and odd global sections of  $\mathcal{O}_T$ :

$$k^{m|n}(T) = (\mathcal{O}(T) \otimes k^{m|n})_0 = \{(t_1, \dots, t_m, \theta_1, \dots, \theta_n) \mid t_i \in \mathcal{O}(T)_0, \theta_j \in \mathcal{O}(T)_1\}$$

## 2.2 Lie Supergroups and SHCP's

A Lie supergroup is a group object in the category of supermanifolds. We are interested in both the real differentiable and the complex analytic category. We shall use the word Lie supergroup whenever the results and definition we give do not depend on the chosen category.

**Definition 2.4.** A *Lie supergroup* (SLG for short) is a supermanifold whose functor of points  $G : (\text{smflds})^o \rightarrow (\text{sets})$  is group valued. Saying that  $G$  is group valued is equivalent to have the following natural transformations satisfying the usual diagrams:

1. Multiplication  $\mu : G \times G \rightarrow G$ , such that  $\mu \circ (\mu \times \text{id}) = (\mu \times \text{id}) \circ \mu$ ,
2. Unit  $e : e_k \rightarrow G$ , such that  $\mu \circ (\text{id} \otimes e) = \mu \circ (e \times \text{id})$ ,
3. Inverse  $i : G \rightarrow G$ , such that  $\mu \circ (i \times \text{id}) = \mu \circ (\text{id} \times i) = e \circ \text{id}$ ,

Morphisms of Lie supergroups are morphisms of the underlying supermanifolds preserving the group structure. We shall denote the category of Lie supergroups with  $(\text{sgrps})$ .

**Example 2.5.** 1. *The General Linear Supergroup.* Let  $\mathrm{GL}(m|n) : (\mathrm{smflds})^o \longrightarrow (\mathrm{sets})$  be the functor such that  $\mathrm{GL}(m|n)(T)$  are the invertible  $m|n \times m|n$  matrices with entries in  $\mathcal{O}(T)$ :

$$\begin{pmatrix} p_{m \times m} & q_{m \times n} \\ r_{n \times m} & s_{n \times n} \end{pmatrix} \quad (1)$$

where the submatrices  $p$  and  $s$  have even entries and  $q$  and  $r$  have odd entries, the definition on the arrows being clear. The invertibility condition implies that  $p$  and  $q$  are ordinary i.e. non super invertible matrices  $\mathrm{GL}(m|n)$  is the functor of points of a supermanifold (i. e. it is representable) whose reduced space is an open set  $U$  in the ordinary space  $k^{m^2+n^2} = M(m|n)_0$  the matrices with invertible diagonal blocks. In fact one can readily check that

$$\mathcal{O}_{\mathrm{GL}(m|n)} = \mathcal{O}_{k^{m^2+n^2}|2mn}|_U$$

(Recall  $k^{m^2+n^2}|2mn} = M(m|n)$ ).

2. *The Special Linear Supergroup.* We define the following subfunctor  $\mathrm{SL}(m|n)$  of  $\mathrm{GL}(m|n)$ :  $\mathrm{SL}(m|n)(T)$  consists of all matrices with Berezinian equal to 1, where

$$\mathrm{Ber} \begin{pmatrix} p_{m \times m} & q_{m \times n} \\ r_{n \times m} & s_{n \times n} \end{pmatrix} = \det(s^{-1}) \det(p - qsr). \quad (2)$$

The functor  $\mathrm{SL}(m|n)$  is representable and it corresponds to a supermanifold that we call *special linear supergroup*, (see [8] Ex. 5.3.14).

**Definition 2.6.** Given a Lie supergroup, its *Lie superalgebra*  $\mathrm{Lie}(G)$  is defined as the set of left invariant vector fields, together with the natural superbracket on them ([8] Ch. 7):

$$[X, Y] = XY - (-1)^{p(X)p(Y)}[Y, X]$$

We recall that a *left invariant vector field* is a vector field  $X$  satisfying the condition:

$$(\mathrm{id} \otimes X) \circ \mu^* = \mu^* \circ X.$$

As in the ordinary case  $\text{Lie}(G)$  can be identified with the tangent space of  $G$  at the identity (which is a topological point i.e. lies in  $\tilde{G}$ ). For more details see [8] Ch. 7, and [43] pg. 276.

We now turn and give a brief description of the equivalent approach to the study of Lie supergroups: the Super Harish-Chandra pairs (SHCP). We are interested both in the analytic and in the differential SHCP's, our main reference for the differential case is [8] Ch. 7, while for analytic one [9] (see also [31] [32]).

**Definition 2.7.** Suppose  $(\tilde{G}, \mathfrak{g})$  are respectively a group (real Lie or complex analytic) and a super Lie algebra. We say  $(\tilde{G}, \mathfrak{g})$  is a *super Harish-Chandra pair* (SHCP for short) if

1.  $\mathfrak{g}_0 \simeq \text{Lie}(\tilde{G})$ ,
2.  $\tilde{G}$  acts on  $\mathfrak{g}$  and this action restricted to  $\mathfrak{g}_0$  is equivalent to the adjoint representation of  $\tilde{G}$  on  $\text{Lie}(\tilde{G})$ . Moreover the differential of such action is the Lie bracket.

A *morphism*  $\psi : (\tilde{G}, \mathfrak{g}) \longrightarrow (\tilde{H}, \mathfrak{h})$  of SHCP's is simply a pair of morphisms  $\psi = (\psi_0, \rho^\psi)$  preserving the SHCP structure that is

1.  $\psi_0 : \tilde{G} \rightarrow \tilde{H}$  is a group morphism;
2.  $\rho^\psi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a super Lie algebra morphism
3.  $\psi_0$  and  $\rho^\psi$  are compatible:

$$\rho^\psi_{|\mathfrak{g}_0} \simeq (d\psi_0)_{1_G} \quad \text{Ad}(\psi_0(g)) \circ \rho^\psi = \rho^\psi \circ \text{Ad}(g)$$

The category of SHCP's is equivalent to the category of SLG's as the next proposition establishes (see [8] 7.4 for the differentiable supermanifolds category and [9, 45] for the complex analytic one).

**Proposition 2.8.** *The functors*

$$\begin{aligned} \mathcal{H} : (\text{sgrps}) &\longrightarrow (\text{shcps}) & \mathcal{K} : (\text{shcps}) &\longrightarrow (\text{sgrps}) \\ G &\longrightarrow (\tilde{G}, \text{Lie}(G)) & (\tilde{G}, \mathfrak{g}) &\longrightarrow (\tilde{G}, \underline{\text{Hom}}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), C_G^\infty)) \end{aligned}$$

*define an equivalence between the two categories of SLG's and SHCP's.*

We now introduce the concept of real structure and real form of a complex analytic Lie supergroup: this will be of fundamental importance for us.

**Definition 2.9.** Let  $G = (\tilde{G}, \mathfrak{g})$  be a complex analytic supergroup. We say that the pair  $(r_0, \rho^r)$  is a *real structure* on  $(\tilde{G}, \mathfrak{g})$  if

1.  $r_0 : \tilde{G} \longrightarrow \tilde{G}$  is a real structure on the ordinary complex group  $\tilde{G}$ , with fixed points being a real Lie group denoted by  $\tilde{G}^r$ . Notice that  $r_0$  is an automorphism of the ordinary real Lie group underlying  $\tilde{G}$ .
2.  $\rho^r : \mathfrak{g} \longrightarrow \mathfrak{g}$  is a  $\mathbb{C}$ -antilinear involutive Lie superalgebra morphism, with its fixed points  $\mathfrak{g}^r = \text{Lie}(\tilde{G}^r)$ .
3.  $(r_0, \rho^r)$  are compatible in the sense of Def. 2.7, that is  $(dr_0)_{1_{\tilde{G}}} = \rho^r$  and  $\rho^r$  intertwines the adjoint action. In other words  $(r_0, \rho^r)$  is an automorphism of  $(\tilde{G}, \mathfrak{g})$  as real Lie supergroup.

We furtherly say that given a real structure  $r = (r_0, \rho^r)$ ,  $G_r = (\tilde{G}^r, \mathfrak{g}^r)$  is a *real form* of  $G = (\tilde{G}, \mathfrak{g})$  and that  $G$  is a *complexification* of  $G_r$ .

If  $G = (\tilde{G}, \mathfrak{g})$  is a complex analytic super Lie group, we can naturally view  $G$  as a real Lie group by considering the  $\tilde{G}$  as an ordinary real Lie group and  $\mathfrak{g}$  as a real Lie superalgebra. In this case we write  $G_{\mathbb{R}}$  and we say that  $G_{\mathbb{R}}$  is the *real supergroup underlying*  $G$ .

We now turn to the functor of points description of real structures and real forms (for more details see [22] Ch. 1). While in general the theory of real structures and real forms of complex supermanifolds is quite involved (see [15] and [11]), for our purpose, since we are interested only in sub-supergroups of the general linear supergroup, the following will suffice.

Let us consider first the complex general linear supergroup  $\text{GL}(m|n)$ . A  $T$ -point  $\phi$  of its underlying real supergroup  $\text{GL}(m|n)_{\mathbb{R}}$  is a morphism

$$\mathcal{O}(\text{GL}(m|n)) \rightarrow \mathcal{O}(T) \otimes \mathbb{C}$$

where  $\mathcal{O}(T)$  denotes the global sections of the sheaf  $\mathcal{O}_T$  for a real supermanifold  $T = (|T|, \mathcal{O}_T)$  and similarly  $\mathcal{O}(\text{GL}(m|n))$  denotes the global sections of the sheaf of the complex supermanifold  $\text{GL}(m|n)$ . More precisely, once we fix



global coordinates  $z_{ij}$  and  $\zeta_{kl}$  corresponding to the diagonal block and off diagonal block matrices respectively, we have that we can effectively represent a morphism  $\phi : T \rightarrow \mathrm{GL}(m|n)_{\mathbb{R}}$  in a matrix form

$$\phi \cong \begin{pmatrix} t & \theta \\ \eta & s \end{pmatrix},$$

where  $t, s$  are invertible matrices with entries in  $\mathcal{O}(T)_0 \otimes \mathbb{C}$ ,  $\theta, \eta$  are matrices with entries in  $\mathcal{O}(T)_1 \otimes \mathbb{C}$  and we have

$$\phi(z_{ij}) = t_{ij} + it'_{ij}, \quad 1 \leq i, j \leq m,$$

$$\phi(z_{ij}) = s_{ij} + is'_{ij}, \quad m+1 \leq i, j \leq m+n,$$

$$\phi(\zeta_{kl}) = \theta_{kl} + i\theta'_{kl}, \quad 1 \leq k \leq m, m+1 \leq l \leq m+n,$$

$$\phi(\zeta_{kl}) = \eta_{kl} + i\eta'_{kl}, \quad 1 \leq l \leq m, m+1 \leq k \leq m+n.$$

Consider now a complex analytic supergroup  $G \subset \mathrm{GL}(m|n)$ , considered as a real supermanifold as explained above. In order to define a real form of  $G$  through the functor of points formalism, we need an involution of  $G_{\mathbb{R}}(T)$  functorial in  $T$ , for all  $T \in (\mathrm{smflds})_{\mathbb{R}}$ . Using this formalism, we can very explicitly define such involutions; let us see a concrete example.

On  $G_{\mathbb{R}}(T)$  we look at the involution:

$$G_{\mathbb{R}}(T) \xrightarrow{\rho_R} G_{\mathbb{R}}(T), \quad \begin{pmatrix} t & \theta \\ \eta & s \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{t} & \bar{\theta} \\ \bar{\eta} & \bar{s} \end{pmatrix}.$$

where  $\bar{t} = t_0 - it_1$  and similarly for  $\bar{\theta}, \bar{\eta}$  and  $\bar{s}$ . Notice that  $\rho_T$  preserves the group multiplication.

$$\rho_T(gg') = \rho_T(g)\rho_T(g'),$$

thus the set of fixed points of this involution is, not just a real supermanifold, but a real supergroup.

## 2.3 Quotients and covering spaces

In this section we want to briefly outline the theory of quotients of supergroups and their covering spaces. For the quotients, our main references is [8] Ch. 9 and [23].

Let  $G$  be a Lie supergroup and  $H$  a closed Lie subsupergroup. Let  $\mathfrak{g} = \text{Lie}(G)$  and  $\mathfrak{h} = \text{Lie}(H)$ . For each  $Z \in \mathfrak{g}$ , let  $D_Z$  be the left invariant vector field on  $G$  defined by  $Z$ . For  $x_0 \in \tilde{G}$  let  $\ell_{x_0}$  and  $r_{x_0}$  be the left and right translations of  $G$  by  $x_0$ . For any subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  we define the subsheaf  $\mathcal{O}_{\mathfrak{k}}$  of  $\mathcal{O}_G$  as:

$$\mathcal{O}_{\mathfrak{k}}(U) = \{f \in \mathcal{O}_G(U) \mid D_Z f = 0 \text{ on } U \text{ for all } Z \in \mathfrak{k}\}.$$

On the other hand, for any open subset  $\tilde{W} \subset \tilde{G}$ , invariant under right translations by elements of  $\tilde{H}$ , we define

$$\mathcal{O}_{inv}(\tilde{W}) = \{f \in \mathcal{O}_G(\tilde{W}) \mid f \text{ is invariant under } r_{x_0} \text{ for all } x_0 \in \tilde{H}\}.$$

If  $\tilde{H}$  is connected we have

$$\mathcal{O}_{inv}(\tilde{W}) = \mathcal{O}_{\mathfrak{h}_0}(\tilde{W})$$

as one can readily check by looking infinitesimally at the condition  $r_{x_0}^* f = f$ , for all  $x_0 \in \tilde{H}$ . For any open set  $\tilde{U}$  in the topological space  $\tilde{X} := \tilde{G}/\tilde{H}$  with  $\tilde{W} = \tilde{\pi}^{-1}(\tilde{U})$ ,  $\tilde{\pi} : \tilde{G} \rightarrow \tilde{X}$ , we define the sheaf  $\mathcal{O}_X$  on  $\tilde{X}$ :

$$\mathcal{O}_X(\tilde{U}) = \mathcal{O}_{inv}(\tilde{W}) \cap \mathcal{O}_{\mathfrak{h}}(\tilde{W}) \subset \mathcal{O}_G(\tilde{W}).$$

Clearly  $\mathcal{O}_X(\tilde{U}) = \mathcal{O}_{\mathfrak{h}}(\tilde{W})$  if  $\tilde{H}$  is connected. The subsheaf  $\mathcal{O}_X$  is a sheaf of superalgebras on  $\tilde{X}$ . We have thus defined a superspace  $X = (\tilde{G}/\tilde{H}, \mathcal{O}_X)$ . This is actually a supermanifold and it is called the *quotient of  $G$  by the closed subsupergroup  $H$*  and denoted with  $G/H$  (See [8] 9.3.7).

**Observation 2.10.** We want to remark that it is possible to develop the theory of quotients using the equivalent functor of points approach. It is important however to notice that in dealing with the quotient of a supergroup  $G$  by a closed subsupergroup  $H$  the functor given by  $T \mapsto G(T)/H(T)$  is not the functor of points of the quotient supermanifold  $G/H$ , but we need to take the *sheafification* of such functor (see [8] Appendix B). Given this extra subtlety to deal with, we have preferred, in order to ease our exposition, to give our proofs mainly in the SHCP's category context, though some of our formulas are more easily interpreted using the functor of points approach.

We now turn to examine actions and homogeneous spaces more in detail.

**Definition 2.11.** We say that a Lie supergroup  $G$  *acts* on a supermanifold  $M$  if there exists a morphism of supermanifolds  $a : G \times M \longrightarrow M$ ,  $a(g, x) := g \cdot x$  for all  $g \in G(T)$ ,  $x \in M(T)$ , where  $T$  is a generic supermanifold, such that:

1.  $1 \cdot x = x$ ,  $\forall x \in M(T)$
2.  $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ ,  $\forall x \in M(T)$ ,  $\forall g_1, g_2 \in G(T)$ .

We say that the action  $a$  is *transitive*, if there exists  $p \in \widetilde{M}$  such that  $a_p : G \rightarrow M$ ,  $a_p(g) := g \cdot p$ , is a surjective submersion. In this case we also say that  $M$  is an *homogeneous superspace* or a  $G$ -*superspace*.

**Observation 2.12.** Let  $g \in \widetilde{G}$ , then  $a_{g \cdot p} = a_p \circ r_g$ , so that if  $a_p$  is submersive for one  $p \in \widetilde{M}$  then it is submersive for all  $p \in \widetilde{M}$ . This allows to prove that  $a$  is transitive if and only if the following two conditions hold:

1.  $\widetilde{a} : \widetilde{G} \times \widetilde{M} \rightarrow \widetilde{M}$  is transitive;
2.  $(da_p)_e : \mathfrak{g} \rightarrow T_p(M)$  is surjective for one  $p$  (hence for all  $p \in \widetilde{M}$ );

We send the reader to [8] to Ch. 9 for more details.

In ordinary geometry we have that if a Lie group  $G$  acts transitively on a manifold  $M$ , if  $G_p$  denotes the stabilizer of the point  $p$ , the quotient  $G/G_p \cong M$ .

In supergeometry the stabilizer of a (topological) point is naturally defined through its functor of points.

**Definition 2.13.** Let  $G$  be a super Lie group acting on a supermanifold  $M$ , and let  $p \in \widetilde{M}$ . We define  $G_p$  the *stabilizer* at  $p$  as the functor from (smflds) to (sets) given by:

$$G_p(T) := \{g \in G(T) \mid g \cdot p_T = p_T\} \subset G(T), \quad \forall T \in (\text{smflds}).$$

(we leave to the reader its definition on morphisms).

The following proposition summarizes all the important results about the stabilizer.

**Proposition 2.14.** *Let  $G$  be a Lie supergroup acting on the supermanifold  $M$  and let  $p \in \widetilde{M}$ .*

1.  $G_p$  is a super Lie subgroup of  $G$ , i. e. the functor of points in Def. 2.13 is representable.

2. Let  $(\widetilde{G}_p, \mathfrak{g}_p)$  be the SHCP associated with the stabilizer  $G_p$ . Then  $\widetilde{G}_p \subseteq \widetilde{G}$  is the classical stabilizer of  $p$  with respect to the reduced action and  $\mathfrak{g}_p = \ker da_p$ .
3. If furthermore  $G$  acts transitively on  $M$  then  $G/G_p \cong M$ .

*Proof.* See [8] Ch. 8, 8.4.7 and Ch. 9, 9.1.4. ■

We now turn to the concept of covering space in the supersetting.

**Definition 2.15.** Let  $M$  and  $N$  be pathwise connected and locally pathwise connected supermanifolds. Suppose  $\pi : M \rightarrow N$  is a morphism such that

1.  $\widetilde{\pi}$  is surjective
2. for each  $x \in \widetilde{N}$  there exists a full open submanifold  $U \subseteq N$ ,  $x \in \widetilde{U}$ , and a family of disjoint open full submanifolds  $V_i$  of  $M$ , such that  $\widetilde{\pi}^{-1}(\widetilde{U})$  is the disjoint union of the open subsets  $\widetilde{V}_i \subset \widetilde{M}$
3. for each  $i$ ,  $\pi|_{V_i} : V_i \rightarrow U$  is a superdiffeomorphism

Then we say that  $(M, \pi)$  is a *covering space* of  $N$ .

**Remark 2.16.** If  $\pi : M \rightarrow N$  is a covering, then  $\widetilde{\pi} : \widetilde{M} \rightarrow \widetilde{N}$  is a covering.

Suppose that  $G$  is a SLG. We denote with  $G_e$  the full subSLG of  $G$  whose reduced space is the identity component of  $G$ .

**Lemma 2.17.** Suppose  $G$  is a complex analytic connected SLG and  $A \subseteq G$  is a closed subSLG of  $G$  then

1. the canonical projection  $\pi : G \rightarrow G/A$  uniquely factors through the map

$$\pi_e : G/A_e \rightarrow G/A$$

Moreover  $\pi_e$  is a covering map.

2. Let  $B$  be another subgroup of  $G$  such that  $B_e = A_e$ . Suppose also that  $G/A$  admits a complex structure. Then  $G/A_e$  and  $G/B$  inherit in a natural way a complex structure.

*Proof.* (1) The existence of the morphism  $\pi_e$  is clear if we notice that a right  $A$ -invariant section over  $G$  is also right  $A_e$ -invariant (refer to the definition of the sheaf of the supermanifold  $G/A$  seen previously). In order to show that  $\pi_e$  is a covering morphism, it is enough to notice that there exists a sequence of elements  $\{g_i\}$  in  $\tilde{G}$  such that  $A = \bigcup_i g_i.A_e$  (the union being disjoint). From this, the result follows easily.

(2) In order to prove this point, let us recall that a complex structure on a supermanifold is the assignment of an atlas  $\{(U_i, h_i)\}$  such that the transition morphisms

$$h_i h_j^{-1}: \mathbb{C}^{p|q} \supseteq U_i \longrightarrow U_j \subseteq \mathbb{C}^{p|q}$$

are super holomorphic. Hence suppose such a complex structure exists on  $G/A$ . We define a complex structure on  $G/A_e$  through a pull-up procedure as follows. If  $(U_i, h_i)$  is a chart over  $G/A$  consider  $\hat{U}_i := \tilde{\pi}_e^{-1}(U_i)$ . By definition  $W$  is the disjoint union of open subsets  $\hat{U}_i^r$  diffeomorphic (when viewed as open full submanifolds) to  $U_i$ . Define the chart

$$\hat{h}_i^r: \hat{U}_i^r \longrightarrow U_i$$

as  $h_i \circ \pi_e|_{\hat{U}_i^r}$ . Since  $\pi_e|_{\hat{U}_i^r}$  is a diffeomorphism, this defines a complex structure over  $G/A_e$ .

In order to define a complex structure over  $G/B$  one has to proceed with a push-down procedure. We leave the details (completely similar to the above) to the reader (extra care is needed only to show that the push-down does not depend on which open set in the covering we choose). ■

We end this section with two results on SLG's that we shall need in the sequel.

We first notice that if  $M$  and  $N$  are supermanifolds and  $\psi: M \rightarrow N$  is a submersion, then  $\tilde{\psi}: \tilde{M} \rightarrow \tilde{N}$  is an open mapping. It is hence well defined the full open submanifold of  $N$  given by

$$\psi(M) := \left( \tilde{\psi}(\tilde{M}), \mathcal{O}_N|_{\tilde{\psi}(\tilde{M})} \right) \quad (3)$$

**Lemma 2.18.** *Let  $M$  be a Lie supergroup (real or complex),  $A_1$  and  $A_2$  Lie subsupergroups with  $\text{Lie}(A_1) + \text{Lie}(A_2) = \text{Lie}(M)$ . Consider the map*

$$\alpha: A_1 \times A_2 \xrightarrow{i_1 \times i_2} M \times M \xrightarrow{\mu} M$$

where  $i_j : A_i \longrightarrow M$  denotes the canonical immersion of  $A_i$  in  $M$  and  $\mu$  is the multiplication of the supergroup  $M$ . Then

1.  $\alpha$  is a submersion and  $A_1 A_2 := \alpha(A_1 \times A_2)$  is an open full submanifold of  $M$  (see Def. 2.1).
2. If  $\widetilde{A_1} \cap \widetilde{A_2} = \{1\}$ , then  $A_1$  and  $A_2$  are closed. Moreover, if  $\text{Lie}(A_1)_1 \cap \text{Lie}(A_2)_1 = \{0\}$ , then  $\alpha$  is an analytic diffeomorphism of  $A_1 \times A_2$  with  $A_1 A_2$ .

*Proof.* We first notice that at the topological point  $(e, e) \in \widetilde{A_1 \times A_2}$  ( $e$  denoting the identity element),

$$(\text{d}\alpha)_{(e,e)}(X_1, X_2) = (\text{d}i_1)_e X_1 + (\text{d}i_2)_e X_2$$

Since  $i_1$  and  $i_2$  are injective immersions and  $\text{Lie}(A_1) + \text{Lie}(A_2) = \text{Lie}(M)$ , we have that  $\alpha$  is a submersion at  $(e, e)$ . Consider now the two actions

$$\beta : (A_1 \times A_2) \times (A_1 \times A_2) \rightarrow (A_1 \times A_2), \quad \gamma : (A_1 \times A_2) \times M \rightarrow M$$

of  $A_1 \times A_2$  on itself and  $M$  respectively, defined, in the functor of points notation, by

$$\beta_T : (a_1, a_2), (b_1, b_2) \longmapsto (a_1 b_1, a_2 b_2)$$

and

$$\gamma_T : (a_1, a_2), m \longmapsto a_1 m a_2$$

respectively, with  $a_i, b_i \in A_i(T)$ ,  $m \in M(T)$ ,  $T \in (\text{smflds})$ . Given a topological point  $(\bar{a}_1, \bar{a}_2) \in \widetilde{A_1 \times A_2}$  define the maps (again in the functor of points notation)

$$\beta_{\bar{a}_1, \bar{a}_2} : A_1 \times A_2 \longrightarrow A_1 \times A_2, \quad \beta_{\bar{a}_1, \bar{a}_2}(a_1, a_2) := (\bar{a}_1 a_1, a_2 \bar{a}_2)$$

$$\gamma_{\bar{a}_1, \bar{a}_2} : M \longrightarrow M \quad \gamma_{\bar{a}_1, \bar{a}_2} g := \bar{a}_1 g \bar{a}_2$$

In other words, if  $i_{\{(\bar{a}_1, \bar{a}_2)\}} : \{(\bar{a}_1, \bar{a}_2)\} \rightarrow A_1 \times A_2$  denotes the embedding of the topological point  $(\bar{a}_1, \bar{a}_2) \in \widetilde{A_1 \times A_2}$  in  $A_1 \times A_2$ , we have

$$\beta_{\bar{a}_1, \bar{a}_2} := \beta \circ (i_{\{(\bar{a}_1, \bar{a}_2)\}} \times \text{id}_{A_1 \times A_2}) \quad \gamma_{\bar{a}_1, \bar{a}_2} := \gamma \circ (i_{\{(\bar{a}_1, \bar{a}_2)\}} \times \text{id}_M)$$

We have

$$\alpha \circ \beta_{\bar{a}_1, \bar{a}_2} = \gamma_{\bar{a}_1, \bar{a}_2} \circ \alpha \tag{4}$$

Passing to the differentials, we obtain

$$(\mathrm{d}\alpha)_{\bar{a}_1, \bar{a}_2} \circ (\mathrm{d}\beta_{\bar{a}_1, \bar{a}_2})_{e, e} = (\mathrm{d}\gamma_{\bar{a}_1, \bar{a}_2})_{e, e} \circ (\mathrm{d}\alpha)_{e, e}$$

Since  $\beta$  and  $\gamma$  are two actions,  $\beta_{\bar{a}_1, \bar{a}_2}$  and  $\gamma_{\bar{a}_1, \bar{a}_2}$  are isomorphisms. Hence also their differentials  $(\mathrm{d}\beta_{\bar{a}_1, \bar{a}_2})_{\bar{a}_1, \bar{a}_2}$  and  $(\mathrm{d}\gamma_{\bar{a}_1, \bar{a}_2})_{\bar{a}_1, \bar{a}_2}$  are isomorphisms. Since  $(\mathrm{d}\alpha)_{e, e}$  is surjective we obtain that  $(\mathrm{d}\alpha)_{\bar{a}_1, \bar{a}_2}$  is surjective for each  $\bar{a}_1, \bar{a}_2$ .

(2) Suppose  $\widetilde{A_1 \cap A_2} = \{1\}$ . We claim that the underlying morphism

$$\tilde{\alpha} : \widetilde{A_1 \times A_2} \rightarrow \widetilde{A_1 A_2}$$

is a diffeomorphism. Since  $\tilde{\alpha}$  is a submersion it is enough to prove that it is bijective. Suppose that

$$\bar{a}_1 \bar{a}_2 = a_1 a_2$$

for some  $a_i, \bar{a}_i \in \widetilde{A_i}$ . Then

$$\bar{a}_1^{-1} a_1 = (a_2 \bar{a}_2^{-1})^{-1}$$

so that  $\bar{a}_1^{-1} a_1 \in \widetilde{A_1}$  and  $a_2 \bar{a}_2^{-1} \in \widetilde{A_2}$  are in  $\widetilde{A_1 \cap A_2} = \{1\}$ , and bijectivity is proved. Now the topological conclusions is entirely classical and rests on the following arguments:

1.  $\tilde{\alpha} : \widetilde{A_1 \times A_2} \rightarrow \widetilde{A_1 A_2}$  is a diffeomorphism. Since  $\widetilde{A_1 \times A_2} \simeq \widetilde{A_1} \times \widetilde{A_2}$  and since  $\widetilde{A_1} \times \{e\}$  is closed in  $\widetilde{A_1} \times \widetilde{A_2}$ ,  $\widetilde{A_1}$  is locally closed in  $M$ .
2. Being  $\widetilde{A_1}$  a locally closed subgroup of  $M$  it is closed. Similarly for  $\widetilde{A_2}$ .

If  $\mathrm{Lie}(A_1)_1 \cap \mathrm{Lie}(A_2)_1 = \{0\}$  then  $(\mathrm{d}\alpha)_{e, e}$  is bijective. Hence  $\alpha$  is a morphisms such that  $\tilde{\alpha}$  is a diffeomorphism and  $(\mathrm{d}\alpha)_{a_1, a_2}$  is an isomorphism (due to eq.4). Hence  $\alpha$  is a superdiffeomorphism (see [31]).  $\blacksquare$

**Lemma 2.19.** *Let  $G$  be a complex Lie supergroup,  $\mathrm{Lie}(G) = \mathfrak{g}$ . Let  $G_r$  be a real form of  $G$ ,  $\mathrm{Lie}(G_r) = \mathfrak{g}_r$ ,  $\mathfrak{g}_r \otimes \mathbb{C} = \mathfrak{g}$  and let  $R_r$  be a closed subgroup of  $G_r$ ,  $\mathrm{Lie}(R_r) = \mathfrak{r}_r$ .*

*Let  $\mathfrak{q}$  be a complex Lie subalgebra of  $\mathfrak{g}$  such that*

- $\mathfrak{g}_r + \mathfrak{q} = \mathfrak{g}$ ;
- $\mathfrak{g}_r \cap \mathfrak{q} = \mathfrak{r}_r$ .

Assume that the subsupergroup  $Q$  defined by  $\mathfrak{q}$  in  $G$  is closed. Let  $R_1$  be the Lie supergroup with support  $\widetilde{Q} \cap \widetilde{G}_r$  and Lie algebra  $\mathfrak{r}_r$ . Then

1.  $G_r Q$  is an open full submanifold in  $G$  (in the sense of def. 2.1);
2.  $G_r/R_1 \cong G_r Q/Q$ ;
3.  $G_r/R_r$  has a  $G_r$ -invariant complex structure.

*Proof.* (1). By our hypothesis  $G_r$  and  $Q$  are subgroups of  $G$  such that  $\mathfrak{g}_r + \mathfrak{q} = \mathfrak{g}$ . Hence by Lemma 2.18 we are done.

(2). By Proposition 2.14 it is enough to show the following:

- (a)  $G_r$  acts transitively on  $G_r Q/Q$ .
- (b) the stabilizer of the previous action at the point  $[Q] \in \widetilde{G_r Q}/Q$  is  $R_1$ .

(a) Due to point (1),  $G_r Q$  is an open full subsupermanifold of  $G$ , and  $G_r Q/Q$  is an open full submanifold of the homogeneous space  $G/Q$ . Clearly  $G_r$  acts on  $G_r Q$ , and on  $G_r Q/Q$ . Let us denote with  $a$  such an action:

$$a: G_r \times G_r Q/Q \longrightarrow G_r Q/Q$$

By definition  $G_r$  acts transitively over  $G_r Q/Q$  if

$$\zeta: G_r \longrightarrow G_r Q/Q,$$

is a surjective submersion, where  $\zeta = a_{[Q]}$  in the notation of Def. 2.11,  $[Q] \in \widetilde{G_r Q}/Q$  being the identity coset.

We want to prove this fact.

Notice that at the level of reduced manifold we have a smooth surjective submersion

$$\widetilde{\zeta}: \widetilde{G_r} \longrightarrow \widetilde{G_r Q}/Q \simeq \widetilde{G_r Q}/\widetilde{Q}$$

by the classical theory. We thus need to prove that the differentials  $(d\zeta)_g$  are surjections for all  $g \in \widetilde{G_r}$ . Since

$$\zeta \circ \ell_g = a_g \circ \zeta$$

we have

$$(d\zeta)_g \circ (d\ell_g)_e = (da_g)_e \circ (d\zeta)_e$$



so it is enough to prove that it is a submersion at the identity  $e \in G_r$ . It follows by an easy computation that  $(d\zeta)_e$  is given by

$$\begin{aligned}\mathfrak{g}_r &\longrightarrow (\mathfrak{g}_r + \mathfrak{q})/\mathfrak{q} \\ X &\longmapsto X \bmod \mathfrak{q}\end{aligned}$$

Since  $(\mathfrak{g}_r + \mathfrak{q})/\mathfrak{q} \simeq \mathfrak{g}_r/(\mathfrak{g}_r \cap \mathfrak{q}) \simeq \mathfrak{g}_r/\mathfrak{r}_r$  we are done.

(b) In order to prove that the stabilizer  $H$  of the action of  $G_r$  on  $G_r Q/Q$  is  $R_1$ , by Prop. 2.14 we shall prove that  $\tilde{H} = \widetilde{R_1}$  and  $\text{Lie}(H) = \mathfrak{r}_r$ .

The stabilizer  $H$  of  $[Q]$  (in the language of SHCP's) is

$$\tilde{H} := \{ g_0 \in \widetilde{G_r} \mid g_0 q \in \tilde{Q} \quad \forall q \in \tilde{Q} \} = \widetilde{G_r \cap Q} = \widetilde{R_1}$$

$$\text{Lie}(H) := \{ X \in \text{Lie}(G) \mid (d\zeta)_e(X) = 0 \} = \ker(d\zeta)_e$$

Hence, it only remains to prove that

$$\ker(d\zeta)_e = \mathfrak{r}_r$$

For this, recall that the sections of the sheaf  $\mathcal{O}_{G_r Q/Q}$  are the sections in  $\mathcal{O}_{G_r Q}$  that are right  $Q$ -invariant (see beginning of Sec. 2.3). In particular

$$\zeta^*(f) = (1 \otimes i_{[Q]}^*) \circ a^*(f) = r_Q^*(f) = f$$

where  $i_{[Q]}^*$  is the pull-back of the embedding  $i_{[Q]} : [Q] \hookrightarrow G_r Q/Q$ . Hence

$$(d\zeta)_e(X)(f) = X(\zeta^*(f))(e) = X(f)(e) = D_X^L(f)(e)$$

From this and from invariance under right  $Q$  action, it is clear that  $\text{Lie}(H) \supseteq \mathfrak{g}_r \cap \mathfrak{q} = \mathfrak{r}_r$ .

To get the other inclusion, let  $X \neq \mathfrak{q}$ . Using local coordinates around the identity one can easily construct the germ of a section  $f$  that is  $Q$  invariant and such that  $D_X^L(f)(e) \neq 0$ . The germ of  $f$  then extends to a section using a partition of unity argument.

We can thus conclude that

$$G_r/R_1 = G_r Q/Q$$

(3)  $G_r Q$  is open in  $G$ , hence the complex structure of  $G$  is trivially inherited by  $G_r Q$ . Hence also  $G_r Q/Q \simeq G_r/R_1$  inherits the complex structure, being the quotient of two complex super Lie groups. Now notice that  $R_1$  and  $R_r$  being closed subgroups of  $G$  with the same super Lie algebra  $\mathfrak{t}_r$  have the same identity component:  $(R_1)_e = (R_r)_e$ . In view of the Lemma 2.17 we can pull-back the complex structure through the covering map  $p_1 : G_r/(R_r)_e \rightarrow G_r/R_1$ . Finally, using the other covering  $p_2 : G_r/(R_r)_e \rightarrow G_r/R_r$  we can push-down the complex structure on  $G_r/(R_r)_e$  and we obtain a complex structure on  $G_r/R_r$ .  $\blacksquare$

## 2.4 Super Vector Bundles and Induced Representations

We want to discuss the basic aspects of the theory of associated super vector bundles.

The following fact is simple (see also [8] Ch. 7, 8, 9).

**Proposition 2.20.** *Let  $G$  be a complex SLG, and  $V = V_0 \oplus V_1$  is a complex finite-dimensional vector superspace. The following notions are equivalent.*

1. *Action of  $G$  on  $V$  according to Def. 2.11*

$$G(\cdot) \times V(\cdot) \rightarrow V(\cdot)$$

*We will refer to this as a  $G$  linear action via the functor of points.*

2. *A morphism of supermanifolds:*

$$a : G \times V \rightarrow V$$

*obeying the usual commutative diagrams and satisfying:*

$$a^*(V^*) \subseteq \mathcal{O}(G) \otimes V^*$$

*We will refer to this as a  $G$  linear action.*

3. *SLG's morphism*

$$G \rightarrow \mathrm{GL}(V)$$

*We will refer to this as a  $G$ -representation.*

4. A natural transformation

$$G(\cdot) \rightarrow \mathrm{GL}(V)(\cdot)$$

We will refer to this as a  $G$ -representation via the functor of points.

5. A SHCP representation, that is:

(a) a Lie group morphism

$$\tilde{\pi} : G_0 \rightarrow \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$$

(b) a super Lie algebra morphism

$$\rho^\pi : \mathfrak{g} \rightarrow \mathrm{End}(V)$$

such that

$$\tilde{\pi}(g)\rho^\pi(X)\tilde{\pi}(g)^{-1} = \rho^\pi(\mathrm{Ad}(g)X), \quad \rho^\pi|_{\mathfrak{g}_0} \simeq d\tilde{\pi}$$

This proposition is fundamental and it will enter in many definitions and propositions. From now on we will often switch between the various equivalent pictures.

**Remark 2.21.** Notice that the first four characterization of the concept of action are merely an application of Yoneda's lemma; the only check concerns the equivalence between any of the first four notion with the fifth one. The above proposition reflects, at the level of representation theory, the equivalence existing between SLGs, SHCPs and the functor of points picture. More details about this equivalence in the analytic setting, which is the one that we are taking into consideration, are found in [9].

Let us now introduce the concept of *contragredient representation*.

**Definition 2.22.** Let  $\pi : G(\cdot) \longrightarrow \mathrm{GL}(V)(\cdot)$  be a  $G$ -representation. As in the classical setting, we have that  $\pi$  induces another representation on  $V^*$  that we call the *contragredient representation*. Such a representation is given by:

$$\pi_c(g)(f)(v) = f(\pi(g^{-1})v)$$

Equivalently if  $\pi = (\tilde{\pi}, \rho^\pi)$  is a SHCP's representation of  $(G_0, \mathfrak{g})$  on  $V$ , the *contragredient representation*  $(\tilde{\pi}_c, \rho_c^\pi)$  with respect to  $(\tilde{\pi}, \rho^\pi)$  is defined as:

$$\tilde{\pi}_c(g)(f)(v) := f(\tilde{\pi}(g^{-1})v), \quad \rho_c^\pi(X)(f)(v) := f(\rho^\pi(-X)v)$$

with  $f \in V^*$ ,  $v \in V$ ,  $g \in G_0$ ,  $X \in \mathfrak{g}$ . Given an action  $a$  of  $G$  in  $V$ , we shall denote the corresponding contragredient action with  $a_c$ .

We now want to define the concept of super vector bundle on  $G/H$  associated to a finite dimensional  $H$ -representation, where  $H$  a closed subSLG of  $G$ . Classically if  $\sigma_0$  is a representation in  $V$  of the ordinary Lie group  $H_0$  a closed subgroup of  $G_0$ , the global sections of the associated bundle consist of the  $H_0$ -covariant functions, that is the functions  $f : G \longrightarrow V$  satisfying:

$$f(gh) = \sigma_0(h)^{-1}f(g) \quad (5)$$

We now want to give this same concept in supergeometry in the three different settings, SLG's, SLG's through the functor of points and SHCP's in the same spirit as in Prop. 2.20. Preliminary to this, let us recall the concept of *super vector bundle* (see, for example, [15]).

**Definition 2.23.** Let  $M = (\widetilde{M}, \mathcal{O}_M)$  be a supermanifold. A *super vector bundle*  $\mathcal{V}$  of rank  $p|q$  is a locally free sheaf of rank  $p|q$  that is for each  $x \in \widetilde{M}$  there exist  $U_x$  open such that  $\mathcal{V}(U) \cong \mathcal{O}_M(U)^{p|q} := \mathcal{O}_M(U) \otimes k^{p|q}$ .  $\mathcal{V}$  is a sheaf of  $\mathcal{O}_M$  modules and at each  $x \in \widetilde{M}$ , the stalk  $\mathcal{V}_x$  is a  $\mathcal{O}_{M,x}$ -module. We define the *fiber* of  $\mathcal{V}$  at the point  $x$  as the vector superspace  $\mathcal{V}_x/m_x\mathcal{V}_x$ , where  $m_x$  is the maximal ideal of  $\mathcal{O}_{M,x}$ .

More explicitly, if  $\mathcal{V}(U) \cong \mathcal{O}_M(U)^{p|q}$ , we have that the stalk at  $x$  is  $\mathcal{V}_x = \mathcal{O}_{M,x}^{p|q}$ , while the fiber is  $k^{p|q}$ .

**Definition 2.24.** Let  $G$  be a SLG,  $H$  a closed subSLG,  $\sigma$  a finite dimensional complex representation of  $H$  on  $V$ , with  $\sigma = (\tilde{\sigma}, \rho^\sigma)$  in the language of SHCP's. Consider the sheaf over  $G_0/H_0$

$$\mathcal{A}(U) := \mathcal{O}_G(p_0^{-1}(U)) \otimes V$$

where  $p : G \rightarrow G/H$  is the canonical submersion.

- We define in the SLG context the assignment:

$$U \longmapsto \mathcal{A}_{SLG}(U) \quad (6)$$

where:

$$\mathcal{A}_{SLG}(U) := \{ f \in \mathcal{A}(U) \mid (\mu_{G,H}^* \otimes 1)(f) = (1 \otimes a_c^*)f \} \quad (7)$$

and

$$\mu_{G,H} : G \times H \xrightarrow{1 \times i} G \times G \xrightarrow{\mu} G$$

$a_c : H \times V^* \rightarrow V^*$  denotes the action associated to the contragredient representation of  $H$  in  $V^*$  with respect to  $\sigma$ .

- We define in the SHCP context the assignment:

$$U \longmapsto \mathcal{A}_{SHCP}(U) \quad (8)$$

where:

$$\mathcal{A}_{SHCP}(U) := \left\{ f \in \mathcal{A}(U) \mid \begin{cases} (r_h^* \otimes 1)f = (1 \otimes \tilde{\sigma}(h)^{-1})(f) & \forall h \in H_0 \\ (D_X^L \otimes 1)f = (1 \otimes \rho^\sigma(-X))f & \forall X \in \mathfrak{h}_1 \end{cases} \right\} \quad (9)$$

- We define in the functor of points context the assignment:

$$U \longmapsto \mathcal{A}_{FOP}(U) \quad (10)$$

where

$$\mathcal{A}_{FOP}(U) := \{ f : p^{-1}(U) \rightarrow V \otimes_k k^{1|1} \mid f_T(gh) = \sigma_T(h)^{-1} f_T(g) \} \quad (11)$$

with

$$\sigma' : H \rightarrow \mathrm{GL}(V \otimes_k k^{1|1}), \quad \sigma' = \sigma \otimes 1$$

where  $p^{-1}(U) \subset G$  is the open subsupermanifold corresponding to the open set  $p_0^{-1}(U)$ .

Notice that also  $\mathcal{A}_{FOP}(U) \subset \mathcal{A}(U)$  since elements of the sheaf  $\mathcal{O}_G(p_0^{-1}(U))$  identify with morphisms of supermanifolds

$$f : p^{-1}(U) \rightarrow V \otimes_k k^{1|1}$$

We now establish the equivalence of the three notions introduced in the previous definition.

**Proposition 2.25.** *The assignments*

$$U \longmapsto \mathcal{A}_{SLG}(U), \quad U \longmapsto \mathcal{A}_{SHCP}(U), \quad U \longmapsto \mathcal{A}_{FOP}(U) \quad (12)$$

*define super vector bundles on  $G/H$  with fiber isomorphic to  $V$ . Moreover we have*

$$\mathcal{A}_{FOP} = \mathcal{A}_{SHCP} = \mathcal{A}_{SLG} \quad (13)$$

*Proof.* We first show that  $\mathcal{A}_{SHCP}$  is a super vector bundle on the quotient  $G/H$ . Let us denote  $\mathcal{A}_{SHCP}$  with  $\mathcal{F}$ . We need to show that  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_{G/H}$ -modules, and that it is locally free.  $\mathcal{O}_{G/H}$  acts naturally on the first component of  $\mathcal{A}$ , we now want to show that such an action is well defined on  $\mathcal{F}$ , so that  $\mathcal{F}(U)$  is an  $\mathcal{O}_{G/H}(U)$ -module for all open  $U$ . Indeed, if  $\phi \in \mathcal{O}_{G/H}(U)$  and  $f \in \mathcal{F}(U)$  then  $(r_h^* \otimes 1)(\phi f) = (1 \otimes \sigma(h))^{-1}(\phi f)$ , and  $(D_X^L \otimes 1)(\phi f) = (1 \otimes \rho(-X))(\phi f)$  (due to the right  $H$  invariance of  $\phi$ ). Moreover it is clear that  $\mathcal{F}$  is a sheaf since, for each open  $U \subseteq G/H$ ,  $\mathcal{F}(U)$  is a sub $\mathcal{O}_{G/H}(U)$ -module of  $\mathcal{O}_G(U) \otimes V$ . Using the fact  $U \mapsto \mathcal{O}_G(U) \otimes V$  is a sheaf over  $G/H$  and the fact that right  $H$ -invariance is a local property, it follows that  $\mathcal{F}$  is a sheaf over  $G/H$ .

In order to prove the local triviality of the sheaf  $\mathcal{F}$ , we will use the existence of local sections for  $p: G \rightarrow G/H$ . In section 2.3 we have seen that we have a local isomorphism:

$$\gamma: W \times H \longrightarrow p^{-1}(W)$$

so that we can define a section:

$$s: W \longrightarrow p^{-1}(W)$$

such that  $s^*(f) = (1 \otimes i_e^*)\gamma^*(f)$ . Notice that  $s$  can also be described as  $\gamma \circ (1 \times i_{\{e\}})$  where  $i_{\{e\}}: \{e\} \rightarrow H$  is the embedding of the topological point  $e$  into  $H$ .

Suppose hence that a neighborhood  $U$  of 1 admitting a local section  $s$  has been fixed. Define the following two maps

$$\begin{aligned} \eta: \mathcal{F}(U) &\longrightarrow \mathcal{O}(U)_{G/H} \otimes V \\ F &\longmapsto f_F := (s^* \otimes 1_V)(F) \end{aligned}$$

and

$$\begin{aligned} \zeta: \mathcal{O}(U)_{G/H} \otimes V &\longrightarrow \mathcal{F}(U) \\ f &\longmapsto F_f := (\gamma^* \otimes 1_V)(1_U \otimes a_c^*)f \end{aligned}$$

It is easy to check that  $\eta$  and  $\zeta$  are one the inverse of the other.

We now go to the equalities:  $A_{SHCP} = A_{SLG} = A_{FOP}$ . The equality  $A_{SHCP} = A_{SLG}$  is proved in [4]. In order to prove  $A_{FOP} = A_{SLG}$ , it is enough to notice that condition (11) is equivalent to the commutativity of the following diagram

$$\begin{array}{ccccc} G \times H & \xrightarrow{\mu} & G & \xrightarrow{f} & V \otimes k^{1|1} \\ \parallel & & & & \parallel \\ G \times H & \xrightarrow{c(f \times 1_H)} & H \times (V \otimes k^{1|1}) & \xrightarrow{\sigma'^{-1}} & V \otimes k^{1|1} \end{array}$$

where  $c: V \times H \rightarrow H \times V$  is the commutation morphism. ■

## 2.5 Fréchet Superspaces and Super Fréchet Representations

In this section we recall few definitions relative to the Fréchet superspaces and representations in Fréchet superspaces. In this particular context the supergeometric theory resembles very closely the ordinary one, we however prefer to recap the key definitions and propositions since the lack of a complete reference.

- Definition 2.26.**
1. We say that a vector superspace  $F = F_0 \oplus F_1$  is a *Fréchet superspace* if each  $F_i$  is a Fréchet space.
  2. Let  $A = A_0 \oplus A_1$  be a Fréchet superspace with respect to the family of seminorms  $\{p_i\}_{i \in I}$ . We say that  $A$  is a *Fréchet superalgebra* if the topology is defined by an equivalent family of submultiplicative seminorms  $\{q_i\}$ :  $q_i(ab) \leq q_i(a)q_i(b)$ .
  3. Suppose now that  $\mathcal{F}$  is a sheaf such that  $\mathcal{F}(U)$  is a Fréchet superalgebra for each  $U$ . We say that  $\mathcal{F}$  is *Fréchet supersheaf* if for each open set  $U$  and for any open cover  $\{U_i\}$  of  $U$  the topology of  $\mathcal{F}(U)$  is the initial topology with respect to the restriction maps  $\mathcal{F}(U) \rightarrow \mathcal{F}(U_i)$ .

The two examples most interesting to us are the sheaves of differential or analytic sections on a supermanifold. The differential setting is treated in detail in [8] Ch. 4; we now briefly discuss the analytic one.

Let  $M = (\widetilde{M}, \mathcal{O}_M)$  be a complex analytic supermanifold. Fix now an open subset  $U \subseteq \widetilde{M}$ . For each compact subset  $K \subset U$  and each differential operator  $D$  over  $U$  define:

$$p_{K,D}(f) := \sup_{x \in K} \left| \widetilde{(D(f))}(x) \right|, \quad f \in \mathcal{O}_M(U).$$

As before, one can readily check that each  $p_{K,D}$  defines a seminorm. The family of seminorms obtained by considering all the differential operators and the compact subsets  $K$  covering  $U$  endows  $\mathcal{O}_M(U)$  of a Hausdorff locally convex topology, where the open sets that form a subbase of zero for the topology are:

$$\{ f \in \mathcal{O}_M(U) \mid p_{K,D}(f) < \epsilon \}$$

for all  $K \subseteq U$  compact,  $D \in \text{Diff}(U)$ , the differential operators on  $U$ , and  $\epsilon > 0$ , together with their finite intersections.

**Remark 2.27.** Unlike the classical setting, in order to topologize the holomorphic structural sheaf we need to consider seminorms containing the differential operators in their definition. This is necessary as the following example shows. Consider the holomorphic manifold  $\mathbb{C}^{1|1} = (\mathbb{C}, \mathcal{H}_{\mathbb{C}} \otimes \wedge_{\mathbb{C}}[\theta])$ . A section  $f$  of  $\mathcal{H}_{\mathbb{C}} \otimes \wedge_{\mathbb{C}}[\theta]$  can be written as

$$f(z) = f_0(z) + f_1(z)\theta.$$

It is clear that, in order to obtain a (Hausdorff) Frechét space we need to consider not only the seminorms  $p_K(f) = \sup_{z \in K} |f_0(z)|$ , but also

$$p_{K, \frac{\partial}{\partial \theta}}(f) = \sup_{z \in K} \left| \frac{\partial}{\partial \theta} f(z) \right| = \sup_{z \in K} |f_1(z)|$$

In complete analogy with the ordinary setting one can prove the following results (see [8] 4.5 and Appendix C for details).

**Lemma 2.28.** *Let  $U$  be a chart with coordinates  $t^i, \theta^j$  and let  $\{f_n\}$  be a sequence in  $\mathcal{O}_M(U)$ :*

$$f_n = \sum_I f_{nI} \theta^I \quad f_{nI} \in C_M^\infty(U).$$



1.  $\{f_n\}$  is a Cauchy sequence in  $\mathcal{O}_M(U)$  if and only if, for each  $I$ , the sequence  $\{f_{nI}\}$  is a Cauchy sequence in  $C_M^\infty(U)$ .
2.  $\mathcal{O}_M(U)$  is complete, moreover it is a Fréchet superalgebra.

**Proposition 2.29.** *Let  $\{U_i\}$  be an open cover of an open set  $U \subset \widetilde{M}$ , let also  $\{s_n\}$  be a sequence in  $\mathcal{O}_M(U)$ .*

1.  $\{s_n\}$  converges to  $s$  in  $\mathcal{O}_M(U)$  if and only if  $\{s_n|_{U_i}\}$  converges to  $s|_{U_i}$  in  $\mathcal{O}_M(U_i)$  for each  $U_i$ ;
2.  $\{s_n\}$  is a Cauchy sequence in  $\mathcal{O}_M(U)$  if and only if  $\{s_n|_{U_i}\}$  is a Cauchy sequence in  $\mathcal{O}_M(U_i)$  for each  $U_i$ ;
3.  $\mathcal{O}_M(U)$  is a Fréchet superalgebra for each open subset  $U$ .

As an immediate consequence we have the following.

**Corollary 2.30.**  $\mathcal{O}_M$  is a Fréchet supersheaf.

Another relevant example of Fréchet supersheaf of importance to us, is the sheaf corresponding to an associated bundle, studied in Sec. 2.4.

**Lemma 2.31.** *Suppose  $\mathcal{F}$  is a sheaf over a topological, second countable, Hausdorff space  $\widetilde{M}$ , and suppose there exists an open cover  $\{U_i\}_{i \in I}$  of  $\widetilde{M}$  such that*

$$\mathcal{F}|_{U_i}$$

*is a Fréchet sheaf for each  $i \in I$ . There exists a unique structure of Fréchet sheaf for  $\mathcal{F}$ .*

*Proof.* Let  $U$  be a generic open subset of  $\widetilde{M}$ . We need to endow  $\mathcal{F}(U)$  with a Fréchet space structure such that the restriction maps are continuous.

As for the topology, we define for each  $i \in I$

$$\widehat{U}_i := U_i \cap U$$

The  $\widehat{U}_i$  form an open cover of  $U$ . Being  $\widetilde{M}$  locally compact and second countable there exists a countable subcover  $\widehat{V}_j$ . By a standard result in the theory of Fréchet spaces, we have that  $\mathcal{F}(U)$ , with the initial topology induced by the family of maps

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\widehat{V}_j),$$

is a Fréchet space.

To end the proof we should verify that

1. for another countable cover, the induced Fréchet structure is the same;
2. restriction maps are continuous.

We leave these checks to the reader. ■

**Proposition 2.32.** *Let  $G$  be a (real or complex) super Lie group and  $H$  a super Lie subgroup. Let  $\sigma = (\sigma, \rho^\sigma)$  be an  $H$ -representation. The sheaf on  $\widetilde{G/H}$ :*

$$U \longmapsto \mathcal{F}(U) = \left\{ f \in \mathcal{O}_G(p_0^{-1}(U)) \mid \left\{ \begin{array}{ll} (r_h^* \otimes 1)f = (1 \otimes \sigma(h)^{-1})(f) & \forall h \in H_0 \\ (D_X^L \otimes 1)f = (1 \otimes \rho^\pi(-X))f & \forall X \in \mathfrak{h}_1 \end{array} \right\} \right\},$$

with  $p_0 : \widetilde{G} \longrightarrow \widetilde{G/H}$ , is a Fréchet supersheaf on  $\widetilde{G/H}$ .

*Proof.* We already showed in Prop. 2.25 that  $\mathcal{F}$  is a line bundle over  $G/H$ , that is locally isomorphic to  $\mathcal{O}_{G/H}$ . Applying Lemma 2.31, we have the result. ■

We now want to define representations of a real or complex analytic supergroup on a Fréchet superspace. We start by recalling the classical notion.

**Definition 2.33.** We say we have a *representation* of a real or complex Lie group  $G_0$  in the complex Fréchet space  $F_0$  if:

1. we have a group homomorphism:

$$\pi_0 : G_0 \longrightarrow \text{Aut}(F_0)$$

where  $\text{Aut}(F_0)$  denotes the set of linear homeomorphisms of  $F_0$  onto itself

2. The map:

$$G_0 \times F_0 \longrightarrow F_0, \quad g, v \longmapsto \pi_0(g)v$$

is continuous.

We further say a vector  $v \in F_0$  is *differentiable* if the continuous map

$$\phi_g : G_0 \longrightarrow F_0, \quad g \longmapsto \pi_0(g)v \tag{14}$$

is differentiable. We denote:

$$C^\infty(\pi_0) := \{v \in F_0 \mid v \text{ differentiable} \}$$

When no confusion is possible we will also denote

$$F_0^\infty := C^\infty(\pi_0)$$

We have that  $F_0^\infty$  is dense in  $F_0$  (see [46] Ch. 5).

The group  $G_0$  acts on  $F_0^\infty$  and this action gives rise to a representation of  $\mathfrak{g}_0 = \text{Lie}(G_0)$  in  $F_0^\infty$  as follows:

$$\begin{aligned} \mathfrak{g}_0 &\longrightarrow \text{End}(F_0^\infty) \\ X &\mapsto d/dt|_{t=0} \phi_{\exp(tX)}(g) \end{aligned}$$

This is called the *infinitesimal representation* associated with  $\pi_0$ .

Item 2. in the previous definition is equivalent to the requirement that  $\pi_0$  is continuous when we endow  $\text{Aut}(F_0)$  with the strong topology, i.e.

$$g_n \xrightarrow{n \rightarrow \infty} g \in G_0 \iff \pi_0(g_n)v \xrightarrow{n \rightarrow \infty} \pi_0(g)v \quad \forall v \in F_0.$$

It is possible to endow  $C^\infty(\pi_0)$  with a finer topology than the one inherited by  $F_0$ , in such a way that it becomes a Fréchet space. This is the relative topology induced by the space of smooth functions  $C^\infty(G_0; F_0)$  through the immersion  $F_0^\infty \hookrightarrow C^\infty(G_0; F_0)$  given by Eq. 14 (see [46]).

**Observation 2.34.** The example to keep in mind is the following. Let  $G_0$  be a Lie group acting on a manifold  $M_0$  and let  $F_0$  be the Fréchet space of continuous functions on  $M_0$  with metric induced by the uniform convergence on compact sets.  $G_0$  has a natural continuous action on such  $F_0$ :  $(g \cdot f)(x) = f(g^{-1}x)$ . In this case  $F_0^\infty$  consists of the differentiable functions, however in order to have completeness we need to require more than just uniform convergence, that is we need uniform convergence of all differentials.

Let us now turn to the supersetting.

**Definition 2.35.** Let  $G$  be a complex or real Lie supergroup. We say we have a *representation* of  $G$  in the complex Fréchet vector superspace  $F$  if:

- $G_0$  acts on  $F = F_0 \oplus F_1$  preserving parity:

$$\pi_0: G_0 \longrightarrow \text{Aut}(F_0) \oplus \text{Aut}(F_1)$$

satisfying Def. 2.33.

- Denote with  $C\text{End}(C^\infty(\pi_0))$  the algebra of continuous linear endomorphisms of  $C^\infty(\pi_0)$  endowed with the Fréchet relative topology inherited from  $C^\infty(G_0; F)$ . There is an even linear map

$$\rho^\pi : \mathfrak{g} \longrightarrow C\text{End}(C^\infty(\pi_0))$$

such that

1.  $\rho^\pi|_{\mathfrak{g}_0} = d\pi_0$
2.  $\rho^\pi([X, Y]) = \rho^\pi(X)\rho^\pi(Y) - (-1)^{|X||Y|}\rho^\pi(Y)\rho^\pi(X)$
3.  $\rho^\pi(\text{Ad}(g)X) = \pi_0(g)\rho^\pi(X)\pi_0(g)^{-1}$

We shall write  $F^\infty = F_0^\infty \oplus F_1^\infty$ .

We notice that  $d\pi_0(X)$  is a well defined operator on  $C^\infty(\pi_0)$ , so that 1. makes sense, with a small abuse of notation.

Let  $G$  be a super Lie group and  $H$  a super Lie subgroup. Let  $\sigma = (\tilde{\sigma}, \rho^\sigma)$  be an  $H$ -representation in the language of SHCP. In Section 2.4 and in Prop. 2.32 we have defined the following Fréchet sheaf on the topological space  $\widetilde{G/H}$ :

$$\mathcal{F}(U) = \left\{ f \in \mathcal{O}_G(p_0^{-1}(U)) \mid \begin{cases} (r_h^* \otimes 1)f = (1 \otimes \tilde{\sigma}(h)^{-1})(f) & \forall h \in \mathfrak{h}_0 \\ (D_X^L \otimes 1)f = (1 \otimes \rho^\sigma(-X))f & \forall X \in \mathfrak{h}_1 \end{cases} \right\}$$

with  $p_0 : \tilde{G} \longrightarrow \widetilde{G/H}$ .

The next proposition is very important in the sequel.

**Proposition 2.36.** *Let the notation be as above and let  $U \subset \widetilde{G/H}$  be a  $G$ -invariant open subset. The assignment:*

1.  $G_0 \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U), \quad g, f \mapsto l_{g^{-1}}^* f$
2.  $\mathfrak{g} \longrightarrow \text{End}(\mathcal{F}(U)), \quad X \mapsto D_{-X}^R$

*gives a representation of  $G$  on the Fréchet superspace  $\mathcal{F}(U)$ , where as usual  $D_{-X}^R = (1 \otimes X)\mu^*$  and the element  $X \in \mathfrak{g}$  is interpreted as a left invariant vector field.*

*Proof.* We first show that  $\mathcal{F}(U)$  is a  $G_0$ -module according to Definition 2.35. This means that if  $\{g_n\}$  is a sequence in  $G_0$  converging to  $g \in G_0$  then  $l_{g_n}^*(f) \rightarrow l_g^*(f)$  in  $\mathcal{F}(U)$ : this is an easy check. Hence it remains to show that  $\mathcal{F}^\infty(U)$  is stable under the action of  $\mathfrak{g}$ . The proof is an immediate consequence of the following lemma.  $\blacksquare$

**Remark 2.37.** If we have a Fréchet representation of  $G$  in  $F$  according to Def. 2.35, we can define a map:

$$G(T) \times F(T) \mapsto F(T), \quad F(T) := (\mathcal{O}(T) \otimes F)_0$$

which is functorial in  $T$  and satisfies the obvious diagrams (compare with Prop. 2.20). If  $F = \mathcal{F}(U)$  as in Prop. 2.36, this action takes the familiar form:

$$(g \cdot f)(x) = f(g^{-1}x), \quad g \in G(T), f \in \mathcal{F}(U)(T), x \in U(T) \quad (15)$$

We may also refer to (15) using the SHCP language.

**Lemma 2.38.** *With the above notation, all vectors in the Fréchet superspace  $\mathcal{F}(U)$  are smooth, in other words:*

$$\mathcal{F}(U) = \mathcal{F}(U)^\infty$$

*Proof.* It is an immediate consequence of item ii in Proposition 4.4.1.7 in [46].  $\blacksquare$

We now turn to examine the decomposition of a super Fréchet representations of a supergroup with respect to the action of an ordinary compact Lie subgroup.

Let  $H$  be a real super Lie group, and  $U$  an ordinary compact Lie subgroup in  $H$ ,  $T \subset U$  a maximal torus. Notice that the following treatment applies also to the case  $T = U$ .

Assume  $H$  acts on a Fréchet superspace  $F$  via the representation  $R$  according to Definition 2.35. Notice that the restriction of the representation  $R$  to  $U$  automatically preserves the  $\mathbf{Z}_2$ -grading  $F = F_0 \oplus F_1$ . Let  $\tau$  be a character of an irreducible representation of  $U$ , that we can assume unitary. We define the operator:

$$P(\tau) = d(\tau) \int_U \tau(k)^{-1} R(k) dk, \quad \text{with} \quad \int_U dk = 1 \quad (16)$$

where  $d(\tau)$  is the degree of  $\tau$  (namely the dimension of the irreducible representation associated with  $\tau$ ). We define  $F(\tau)$  as the closed subspace of  $F$  stable under  $U$  and on which  $U$  acts according to the irreducible representation with character  $\tau$ .  $F(\tau)$  is called the *isotypic subspace* corresponding to the character  $\tau$ .

We stress that, in the whole section,  $F^\infty = F_0^\infty \oplus F_1^\infty$  denotes the space of smooth vectors for the representation  $R$  of  $H$ . When we want to consider smooth vectors for the restriction of  $R$  to a subgroup  $U$  of  $G$  we will add a subscript  $F_U^\infty$ . Clearly, one has  $F^\infty \subseteq F_U^\infty$ .

**Proposition 2.39.** *Let  $R$  be a representation of the compact Lie group  $U$  on the Frechét superspace  $F = F_0 \oplus F_1$ . Then:*

1. *the operator  $P(\tau)$  defined by (16), is an even continuous projection onto the isotypic subspace  $F(\tau) = F(\tau)_0 \oplus F(\tau)_1$ .*
2.  *$F(\tau)$  is a closed subsuperspace of  $F$  and it consists of the algebraic sum of the linear subspaces on which  $U$  acts irreducibly according to the (irreducible) representation with character  $\tau$ . Furthermore the  $F(\tau)$  are linearly independent.*
3.  *$P(\tau)P(\tau') = 0$ , if  $\tau \neq \tau'$ .*
4.  *$P(\tau)$  commutes with the  $U$  action and with any continuous endomorphism of  $F$  commuting with  $U$ .*
5. *on the space of smooth vectors we have  $\sum P(\tau) = \text{id}_{|_{F^\infty}}$ , that is any  $f \in F^\infty$  is expressed as  $\sum_\tau f_\tau$ , which is called the Fourier series of  $f$ . Furthermore, such series converges uniformly.*
6. *Let  $F^0 := \sum F(\tau)$  (algebraic sum). Then  $F^0 \subset F^\infty$  and both are dense in  $F$ .*

*Proof.* Since the representation  $R$  and the projections  $P(\tau)$  are even operators (hence preserving the  $\mathbf{Z}_2$ -grading of  $F$ ), all statements follow from the classical theory (for their proof in the classical setting see, for example, [46]). ■

When necessary we shall stress the fact that the decomposition of  $F$  is under the  $U$ -action by writing  $F_U^0$  and  $F_U(\tau)$ , similarly we write  $P_U(\tau)$  for the operator defined in (16).

Next definition is at the heart of Harish-Chandra approach to the representation theory of semisimple Lie groups.

**Definition 2.40.** We say that a representation  $R$  as above is  $U$ -finite if each  $F(\tau)$  is finite dimensional.

**Lemma 2.41.** *Let the notation and setting be as above.*

1. *Let  $\tilde{F}^0 = \sum_{\tau} L_{\tau}$  be a dense subspace in  $F$ , where the sum is algebraic, the subspaces  $L_{\tau}$  are all finite dimensional and  $L_{\tau} \subset F_U(\tau)$ . Then  $L_{\tau} = F_U(\tau)$  for all  $\tau$  and  $\tilde{F}^0 = F_U^0 \subset F^{\infty}$ . Hence,  $F$  is  $U$ -finite.*
2. *Let  $U'$  be a compact subgroup of  $U$  and assume that  $F$  is  $U'$ -finite. Then  $F$  is also  $U$ -finite and  $F_U^0 = F_{U'}^0$ .*

*Proof.* (1). Since  $P_U(\tau)$  is continuous, we have  $L_{\tau} = P_U(\tau)\tilde{F}^0$  dense in  $F_U(\tau)$  and since the  $L_{\tau}$ 's are finite dimensional  $L_{\tau} = F_U(\tau)$  for all  $\tau$ , consequently  $F$  is  $U$ -finite. Consider now  $F^{\infty}(\tau) := P(\tau)F^{\infty}$ . Since  $F^{\infty}$  is dense in  $F$ ,  $F^{\infty}(\tau)$  is dense in  $F(\tau)$ . Since  $F(\tau)$  is finite dimensional,  $F^{\infty}(\tau) = F(\tau)$ . Hence  $\tilde{F}^0 = F_U^0 \subset F^{\infty}$

(2). Let  $t$  be an irreducible representation of  $U$  with character  $\tau$  (it will be a direct summand in the  $F(\tau)$ ). and let  $s_1 \dots s_r$  be the irreducible inequivalent representations of  $U'$  with characters  $\sigma_1 \dots \sigma_r$  appearing in  $t|_{U'}$ . We have  $F_U(\tau) \subset \sum F_{U'}(\sigma_i)$  because for  $v \in F_U(\tau)$ , and  $h \in U'$

$$R(h)v = t(h)v = t(h) \sum_i P(\sigma_i)v = \sum_i s_i(h)v_i, \quad v_i = P(\sigma_i)v \in F_{U'}(\sigma_i)$$

Consequently  $F_U(\tau)$  are all finite dimensional since by hypothesis we have  $\dim(F_{U'}(\sigma_i)) < \infty$ . This also shows  $F_U^0 \subset F_{U'}^0$ . We now want to show the other way around. By the previous point, we have that since  $F_{U'}(\sigma) \subset F^{\infty}$  any  $f \in F_{U'}(\sigma)$  admits a Fourier expansion,  $f = \sum f_{\tau}$ ,  $\tau$  a character of  $U$ . Hence it is enough to show  $P_U(\tau)F_{U'}(\sigma) = 0$  for almost all  $\tau$ . If  $P_U(\tau)F_{U'}(\sigma) \neq 0$  for a given  $\tau$  we have:

$$0 \neq P_U(\tau)F_{U'}(\sigma) \subset F_U(\tau) \cap F_{U'}(\sigma) \subset F_U(\tau)$$

since  $P_U(\tau)$  commutes with the representation. If this were happening for infinitely many  $\tau$ 's, since  $F_U(\tau)$  are all linearly independent due to Prop. 2.39, we would have  $F_{U'}(\sigma)$  infinite dimensional and this gives a contradiction. ■

### 3 Representations of the Supergroup

The objective of this section is to construct representations of a real supergroup  $G_r$  which correspond infinitesimally to the highest weight Harish-Chandra modules constructed in [10].

Let  $G$  and  $G_r$  be respectively a complex and a real Lie supergroup with  $G$  the complexification of  $G_r$ ,  $G$  simply connected.

Let  $\mathfrak{g}$  and  $\mathfrak{g}_r$  be the Lie superalgebras of  $G$  and  $G_r$  respectively. Assume  $\mathfrak{g}$  and  $\mathfrak{g}_r$  are basic classical Lie superalgebras,  $\mathfrak{g} \neq A(n, n)$ ,  $\mathfrak{g}_1 \neq 0$ , hence we will consider the following list of Lie superalgebras:

$$A(m, n) \text{ with } m \neq n, B(m, n), C(n), D(m, n), D(2, 1; \alpha), F(4), G(3) \quad (17)$$

(see [28] Prop. 1.1).

**Remark 3.1.** The supergroup  $G$  is determined by the SHCP  $(G_{\text{red}}, \mathfrak{g})$ , where  $G_{\text{red}}$  is the reduced ordinary complex Lie group (see Sec. 2.1). By the ordinary theory, we know that since  $\mathfrak{g}_0$  is either semisimple or with a one-dimensional center,  $G_{\text{red}}$  is a matrix Lie and algebraic group.  $(G_{\text{red}}, \mathfrak{g})$  can be then viewed either as an algebraic or an analytic SHCP. As algebraic supergroup it is embedded into a  $GL(V)$  for a suitable superspace  $V$  (see [8] Ch. 11). By [17] the algebraic supergroup  $(G_{\text{red}}, \mathfrak{g})$  has a unique complex analytic supergroup structure which is  $(G_{\text{red}}, \mathfrak{g})$  as analytic SHCP. So  $G$  is a matrix analytic supergroup and consequently also  $G_r$  is a matrix real Lie supergroup, i.e. a closed subsupergroup of the real general linear supergroup.

Fix  $\mathfrak{h}$  and  $\mathfrak{h}_r$  CSA of  $\mathfrak{g}$  and  $\mathfrak{g}_r$  respectively,  $\mathfrak{h}$  the complexification of  $\mathfrak{h}_r$ .  $K_r = (K_r)_0$  is the maximal compact in  $G_{r,0}$ ,  $A_r = (A_r)_0$  the (ordinary) torus,  $A_r \subset G_r$ , while  $\mathfrak{k}_r$ ,  $\mathfrak{h}_r$  the respective Lie superalgebras. We drop the index  $r$  to mean the complexifications. We assume:

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}.$$

Hence our CSA  $\mathfrak{h} = \mathfrak{h}_0$ . Let  $\Delta$  be the root system corresponding to  $(\mathfrak{g}, \mathfrak{h})$  and fix  $P$  a positive system.

Let us define  $\mathfrak{b}^\pm$  and  $\mathfrak{n}^\pm$  the *borel and nilpotent subalgebras*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{b}^+ := \mathfrak{h} \oplus \sum_{\alpha \in P} \mathfrak{g}_\alpha, \quad \mathfrak{n}^+ := \sum_{\alpha \in P} \mathfrak{g}_\alpha.$$



We will call  $B^\pm$  *borel subgroup* and  $N^\pm$  *unipotent subgroup*, their corresponding Lie supergroups in  $G$ .

### 3.1 Hermitian Superspaces

We are interested in quotients of a real supergroup  $G_r$  as above, which admit a complex structure invariant with respect to the natural supergroup action.

**Definition 3.2.** Suppose  $H_r$  is a closed purely even subgroup of  $G_r$ , such that  $|G_r|/|H_r|$  is an hermitian space in the ordinary sense. We say that  $G_r/H_r$  is an *hermitian superspace* if it is a complex supermanifold such that

1. the left translation morphism

$$\rho_g: G_r/H_r \longrightarrow G_r/H_r \quad x \longmapsto gx$$

is a complex supermanifold morphism for all  $g \in |G_r|$ .

2. The almost complex structure  $J_x: T_x M \longrightarrow T_x M$  at each  $x \in |M|$  is  $G$ -equivariant:

$$J_{gx} \circ (d\rho_g)_x = (d\rho_g)_x \circ J_x, \quad \text{for all } x \in |M| \text{ and } g \in |G_r|. \quad (18)$$

We notice that once equation (18) is satisfied, then we can easily recover  $J_{gx}$  if we know  $J_x$ . In fact

$$J_{gH_r} = (d\ell_g)_{H_r} \circ J_{H_r} \circ ((d\ell_g)_{H_r})^{-1}$$

where  $H_r$  is the identity coset in the quotient  $|G_r/H_r|$ . This is a well posed definition if and only if

$$J_{H_r} \circ (d\ell_h)_{H_r} = (d\ell_h)_{H_r} \circ J_{H_r}, \quad \forall h \in |H_r|. \quad (19)$$

Since  $(d\ell_h)_{H_r}$  is the adjoint morphism, we can reformulate equation (19) as:

$$J_{H_r} \circ \text{ad}(X)|_{T_{H_r}M} = \text{ad}(X)|_{T_{H_r}M} \circ J_{H_r}, \quad \forall X \in \text{Lie}(H_r), \quad (20)$$

with  $M = G_r/H_r$ .

The following proposition will turn to be very important in our treatment.

**Proposition 3.3.** *Let  $G_r$  be a real form of the complex supergroup  $G$ . Let us consider  $G/H$ , with  $H$  as above (i.e. the complexification of the purely even subgroup  $H_r$  of  $G_r$ ). Let  $U \subset G/H$  be an open real submanifold of  $G/H$  and further assume  $U$  is an homogeneous space for  $G_r$ . Then  $U$  is an hermitian superspace.*

*Proof.* The fact  $U$  has a complex structure is immediate, since  $U$  is open in  $G/H$ . Such complex structure will also immediately satisfy the conditions in Def. 3.2. ■

## 3.2 The exponential of a nilpotent Lie superalgebra

We want to define the exponential diffeomorphism:

$$\exp : \mathfrak{n}^- \longrightarrow N^-.$$

Everything we will say holds replacing  $-$  with  $+$ , hence to ease the notation we shall drop the index “ $-$ ”.

Let as usual  $k$  be our ground field,  $k = \mathbb{R}$  or  $k = \mathbb{C}$ .

Our purpose in the construction of the exponential diffeomorphism is to obtain *global coordinates* on the nilpotent supergroup  $N$ ; such coordinates are going to be essential for our subsequent treatment.

We start with some general remarks on the functor of the  $\Lambda$ -points, we invite the reader to consult [6] and [7] for the complete details.

Let  $M$  be a supermanifold. Instead of looking at the whole functor of points  $M(\cdot) : (\text{smflds}) \rightarrow (\text{sets})$ , it is sometimes convenient to restrict the functor of points from the category (smflds) to the subcategory (spts) consisting of just the *superpoints*

$$k^{0|n}$$

These are the supermanifolds  $(\{*\}, \Lambda^n)$ , where  $\Lambda^n$  denotes the Grassmann algebra in  $n$  generators over  $k$ . In this approach the set  $M(k^{0|n})$  can be endowed with the structure of an ordinary manifold, but with some peculiarities. The tangent space at a point is a  $\Lambda_0^n$ -module and the change of coordinates induced by a change of coordinates in  $M$  must have  $\Lambda_0^n$ -linear differential. These are called  $\Lambda_0$ -manifolds and we denote with  $(\Lambda_0\text{mflds})$  the corresponding category. The functor

$$(\text{smflds}) \longrightarrow (\Lambda_0\text{mflds}) \quad k^{0|n} \longmapsto M(k^{0|n})$$

is a full and faithful embedding (see [6] Sec. 4, Theorem 4.5). We notice that if  $V$  is a vector superspace we have the identification

$$V(k^{0|n}) \simeq (V \otimes \Lambda^n)_0$$

and the previous result is known as the even rules principle.

If  $M = G$  is a matrix Lie supergroup, the  $\Lambda_0$ -manifold  $G(k^{0|n})$  is a group object in the category  $(\Lambda_0\text{mflds})$  and in particular a Lie group. Hence we can define for each  $k^{0|n}$  the exponential

$$\exp_{k^{0|n}} : \mathfrak{g}(k^{0|n}) \longrightarrow G(k^{0|n})$$

It is easy to check that the differential of this map is  $\Lambda_0$ -linear and that the correspondence  $\mathfrak{g}(k^{0|n}) \rightarrow G(k^{0|n})$  is functorial in  $k^{0|n}$ . Hence we get a morphism

$$\exp : \mathfrak{g} \longrightarrow G$$

In case  $G$  is a nilpotent Lie supergroup, each  $G(\Lambda)$  is a nilpotent Lie group and, by a classical result, each  $\exp_{k^{0|n}}$  is a diffeomorphism. Hence  $\exp$  is a superdiffeomorphism.

### 3.3 Maximal torus and big cell in Lie supergroups of classical type

In this section we want to study the connected ordinary Lie group  $A \subset G$ , called a *maximal torus* of  $G$ , with associated Lie algebra  $\text{Lie}(A) = \mathfrak{h}$  the CSA of  $\mathfrak{g}$ , and its relation with the supergroups  $N^\pm$ . In particular we introduce the supermanifold  $\Gamma := N^-AN^+ \subset G$ , called the *big cell*, which plays a key role in what follows.

**Proposition 3.4.** *The (ordinary) torus  $A$  normalizes  $N^\pm$ .*

*Proof.* We give the proof for  $N^+ = N$ . We want to prove that the conjugation

$$\text{conj}(a) : G \longrightarrow G \quad \text{conj}(a) = \ell_{a^{-1}} \circ r_a, \quad a \in |A|$$

stabilizes  $N$ . Since  $N$  is connected and the exponential map  $\exp : \mathfrak{n} \rightarrow N$  is surjective it is enough to prove that  $(\text{dconj}(a))_1(\mathfrak{n}) \subseteq \mathfrak{n}$ . We know from the infinitesimal theory that  $\text{ad}(\mathfrak{h})(\mathfrak{n}) = \mathfrak{n}$ . Hence, we have

$$\text{Ad}(e^{tX})Y = e^{t\text{ad}X}(Y) \quad \forall X \in \mathfrak{h}, Y \in \mathfrak{n}$$

so that  $\text{Ad}(e^{tX})\mathfrak{n} = \mathfrak{n}$ . Since the exponential map of an abelian connected Lie group is surjective we have that  $\text{Ad}(A)\mathfrak{n} = \mathfrak{n}$ . ■

Next proposition gives a property of real forms of  $\mathfrak{g}$  we will often use.

**Proposition 3.5.** *Let  $\mathfrak{m}_r$  be a real form of the Lie superalgebra  $\mathfrak{g} = \text{Lie}(G)$  containing  $\mathfrak{h}_r$  the CSA of  $\mathfrak{g}_r$ . Then:  $\mathfrak{m}_r + \mathfrak{n}^+ = \mathfrak{g}$  and in particular  $M_r B^+$  is open and full in  $G$ , where  $M_r$  is the connected subgroup of  $G_r$  determined by  $\mathfrak{m}_r$ .*

*Proof.* Since  $\mathfrak{m}_r$  is a real form of  $\mathfrak{g}$ , we have  $\mathfrak{g} = \mathfrak{m}_r \oplus i\mathfrak{m}_r$ . This is equivalent to say that there exists an antilinear involution  $\tilde{\cdot} : \mathfrak{g} \rightarrow \mathfrak{g}$  whose set of fixed points is  $\mathfrak{m}_r$ . Moreover, since  $\mathfrak{h}_r$  is contained in  $\mathfrak{k}_r$  we have that all the roots are imaginary when restricted to  $\mathfrak{h}_r$ . These facts imply that  $\mathfrak{g}_\alpha \tilde{\cdot} = \mathfrak{g}_{-\alpha}$ .

In order to prove our statement it is enough to show that  $X_{-\alpha}$  and  $iX_{-\alpha}$  belong to  $\mathfrak{m}_r + \mathfrak{n}^+$ . We have that:

$$X_{-\alpha} = X_\alpha \tilde{\cdot} = (X_\alpha + X_\alpha \tilde{\cdot}) - X_\alpha \in \mathfrak{m}_r + \mathfrak{n}^+$$

$$iX_{-\alpha} = -iX_\alpha \tilde{\cdot} = (-iX_\alpha \tilde{\cdot} + iX_\alpha) - iX_\alpha \in \mathfrak{m}_r + \mathfrak{n}^+$$
■

**Proposition 3.6.** *Let the notation be as above. Then we have that:*

- 1)  $\tilde{\Gamma} := \widetilde{N^- A N^+}$  is open in  $\tilde{G}$ .
- 2)  $\tilde{A}$ ,  $\tilde{N}^\pm$  are closed and

$$\begin{aligned} \widetilde{N^\pm} \cap \tilde{A} &= \{1\} \\ \text{Lie}(N^\pm) \cap \text{Lie}(A) &= \{0\} \end{aligned}$$

- 3) *The morphism*

$$N^- \times A \times N^+ \longrightarrow G, \quad (n^-, h, n^+) \longmapsto n^- h n^+$$

*is an analytic diffeomorphism onto its image  $N^- A N^+$  which is an open full submanifold of  $G$ .*

*Proof.* (1) is a statement of ordinary geometry.

(2) The first statement is topological, while the second one is a simple check.

(3) Consider the morphism  $\phi : A \times N^+ \longrightarrow AN^+ \subset G$ .  $AN^+$  is a Lie

supergroup, since  $N^+$  is normalized by  $A$  and  $\text{Lie}(AN^+) = \mathfrak{h} + \mathfrak{n}^+$ . Hence  $\phi$  is a diffeomorphism onto its image (apply Lemma 2.18). We now apply again Lemma 2.18 to the map  $\psi : N^- \times AN^+ \rightarrow G$ .  $\psi$  is a diffeomorphism onto its image, which is an open full submanifold of  $G$ .  $\blacksquare$

**Remark 3.7.** In the course of the proof we have seen that the image of the multiplication morphism  $A \times N^+ \rightarrow AN^+ \subset G$  is a Lie supergroup. Since its reduced space is  $\widetilde{B}^+ = \widetilde{AN}^+$  and its Lie superalgebra  $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ , by the SHCP's theory, we have that  $B^+ = AN^+$  (and similarly  $B^- = AN^-$ ), where  $B^+$  is the unique connected subgroup of  $G$  with Lie superalgebra  $\mathfrak{b}^+$ .

We now turn to an important definition: the *big cell* in  $G$ .

**Definition 3.8.** Let  $G$  a Lie supergroup of classical type,  $\text{Lie}(G) = \mathfrak{g}$ ,  $A$ ,  $N^\pm$  as above. We define the *big cell* in  $G$  as open full subsupermanifold of  $G$ :

$$\Gamma := N^- AN^+ \subset G$$

Its underlying topological space  $\widetilde{\Gamma} = \widetilde{N^- AN^+}$  is open and dense in  $G$ .

**Proposition 3.9.** *Let the notation be as above. Then we have that:*

- 1)  $G_r B^\pm$  are full open submanifolds in  $G$ .
- 2)  $G_r/A_r \cong G_r B^\pm/B^\pm$  acquires a  $G_r$  invariant complex structure, hence it is an hermitian superspace (see Sec. 3.1).
- 3)  $N^-$  is a section for  $\Gamma/B^+$ , the left action of  $A$  reads:

$$A \times \Gamma/B^+ \longrightarrow \Gamma/B^+, \quad (h, nB^+) \longmapsto hnh^{-1}B^+$$

*Proof.* (1) Due to Prop 2.18,  $\mathfrak{g}_r + \mathfrak{b}^\pm = \mathfrak{g}$ , hence the map  $\alpha : G_r \times B^\pm \rightarrow G$  is a submersion and the submanifolds  $G_r B^\pm$  are full and open in  $G$ .

(2) If we prove that  $\mathfrak{g}_r \cap \mathfrak{b}^\pm = \mathfrak{h}_r$  and  $\widetilde{G}_r \cap \widetilde{B}^+ = \widetilde{A}_r$  we can apply Lemma 2.19 and conclude. Let us hence proceed to prove these facts. We have

$$\mathfrak{g}_r \cap \mathfrak{b}^\pm = \mathfrak{h}_r.$$

Indeed, let  $X \mapsto \widetilde{X}$  be the conjugation of  $\mathfrak{g}$  associated with  $\mathfrak{g}_r$  as described in the proof of Prop 3.5. Consider the case of  $\mathfrak{b}^+$  for definiteness.  $X \in \mathfrak{b}^+$  can be written as  $X = \sum c_i H_i + \sum_{\alpha \in \Delta^+} d_\alpha X_\alpha$  with  $H_i \in \mathfrak{h}_r$ ,  $X_\alpha \in \mathfrak{g}_\alpha$ , and  $c_i, d_\alpha \in \mathbb{C}$ . Then  $\widetilde{X} = \sum_i \bar{c}_i H_i + \sum_{\alpha \in \Delta^+} \bar{d}_\alpha X_\alpha$ . Since  $\mathfrak{g}_\alpha^\sim = \mathfrak{g}_{-\alpha}$  (see the proof of Prop 3.5), we have that  $\widetilde{X} = X$  if and only if  $X \in \mathfrak{h}_r$ .

Since by assumption the SLG  $G$  is simply connected, there exists an anti-automorphism  $\sigma: G \rightarrow G$  such that  $(d\sigma)_e(X) = \widetilde{X}$ . By the ordinary theory we have that  $\widetilde{\widetilde{G_r}} \cap \widetilde{\widetilde{B^+}} = \widetilde{\widetilde{A_r}}$ . Indeed, let  $a = hn^+ \in \widetilde{\widetilde{G_r}} \cap \widetilde{\widetilde{B^+}}$ , with  $h \in \widetilde{\widetilde{A_r}}$  and  $n^+ \in \widetilde{\widetilde{B^+}}$ . Hence  $hn^+ = \widetilde{\sigma}(h)n^-$  (where  $n^- := \widetilde{\sigma}(n^+)$ ), that is  $\widetilde{\sigma}(h)^{-1}hn^+ = n^-$ , so that  $h = \widetilde{\sigma}(h)$ ,  $n^+ = n^- = 1$ , so  $a \in \widetilde{\widetilde{A_r}}$ .

(3) Since the big cell  $\Gamma \subset G$  is right  $B^+$ -invariant and open, and the canonical projection  $p: G \rightarrow G/B^+$  is a submersion, we can define the open full submanifold of  $G/B^+$ :

$$\Gamma/B^+ := (\widetilde{\Gamma/B^+}, \mathcal{O}_{G/B^+}|_{\widetilde{\Gamma/B^+}})$$

Clearly  $p^{-1}(\Gamma/B^+) = \Gamma$ . We are going to construct a section

$$s: \Gamma/B^+ \longrightarrow \Gamma$$

The local splitting

$$\gamma: N^- \times B^+ \longrightarrow \Gamma$$

is an holomorphic morphism such that  $\gamma^* \mathcal{O}_{\Gamma/B^+} = \mathcal{O}_{N^-} \otimes 1$ . Hence we have an isomorphism  $N^- \rightarrow \Gamma/B^+$  given by the composition of the “canonical” embedding  $i: N^- \hookrightarrow N^- \times B^+$  with  $\gamma$  and  $p$  (which is essentially the same as considering  $p \circ \gamma|_{N^- \times \{e\}}$ ). Its inverse is the required section.  $\blacksquare$

### 3.4 Line bundles on $G/B$

Let us consider a character  $\chi_r$  of the classical real maximal torus  $A_r$  inside the real supergroup  $G_r$ . This character uniquely extends to an holomorphic character of  $A$  and has the form

$$\begin{aligned} \chi: A &\longrightarrow \mathbb{C}^\times \\ \exp(H) &\mapsto e^{\lambda(H)} \end{aligned}$$

for an integral weight  $\lambda \in \mathfrak{h}^*$  (i.e. a weight such that  $\lambda(H_\gamma) \in \mathbb{Z}$  for all roots  $\gamma$ ).

We can trivially extend the character  $\chi$  of  $A$  to a character of the borel subgroup  $B = B^+$ . In fact, since we know by Prop. 3.6 that  $B = AN^+ \simeq$

$A \times N^+$ , we can define  $\chi_B := \chi_A \times 1_{N^+}$ , or, using the functor of points notation:

$$\begin{aligned}\chi_B: B(T) = A(T)N^+(T) &\longrightarrow \mathbb{C}^\times(T) \\ an &\longmapsto \chi(a)\end{aligned}$$

We shall drop the index  $B$  and denote with  $\chi$  both the character of  $A$  and the character of  $B$ . As usual we denote with  $(\chi_0, \lambda)$  the corresponding representation in the SHCP formalism.

The character  $\chi = e^\lambda$  defines according to 2.4 an holomorphic line bundle on  $\widetilde{G/B}$  that we denote with  $L^\chi$  or  $L_\lambda$  depending on the convenience. If  $p: G \longrightarrow G/B$  we have:

$$L^\chi(U) = L_\lambda(U) = \{ f \in \mathcal{O}_G(p^{-1}(U)) \mid \begin{cases} r_h^* f = \chi_0(h)^{-1}(f) & \forall h \in \mathfrak{h}_0 \\ D_X^L f = \lambda(-X)f & \forall X \in \mathfrak{h}_1 \end{cases} \} \quad (21)$$

We can equivalently write:

$$\begin{aligned}L^\chi(U) &= L_\lambda(U) = \{ f : p^{-1}(U) \rightarrow \mathbb{C}^{1|1} \mid \\ &\quad f_T(gb) = \chi_T(b)^{-1} f_T(g) \} \end{aligned}$$

We now turn our attention to the Frechét superspace  $F := L^\chi(\widetilde{\Gamma/B^+})$ , where  $\Gamma = N^-AN^+$  is the big cell in the complex supergroup  $G$ .  $\Gamma$  is neither stable under  $G$ -action nor under the  $G_r$ -action, however, as any neighbourhood of 1, it is stable under the action of  $\mathcal{U}(\mathfrak{g})$  and we want to study such representation. Before doing this we establish suitable coordinates on  $\Gamma$ .

**Proposition 3.10.** *The restriction of the holomorphic line bundle  $L^\chi$  to  $\widetilde{\Gamma/B^+} = \widetilde{N^-AN^+}/B^+$  is trivial:*

$$L^\chi(\widetilde{\Gamma/B^+}) \simeq \mathcal{O}_{G/B}(\widetilde{\Gamma/B^+})$$

*In particular, there is a canonical identification between:*

$$F = L^\chi(\widetilde{\Gamma/B^+}) \simeq \text{Hol}(N^-) \quad (22)$$

*between the sections on the big cell of the line bundle  $L^\chi$  and the holomorphic functions on  $N^-$ .*

*Proof.* In order to prove the triviality of the line bundle  $L^\chi$  over  $\widetilde{\Gamma/B^+}$ , we have to construct a section

$$s: \Gamma/B^+ \longrightarrow \Gamma$$

(see Sec. 2.3). This is the content of Prop. 3.9, (3).

We have maps  $\eta: L^\chi(\widetilde{\Gamma/B^+}) \rightarrow Hol(N^-)$  and  $\zeta: Hol(N^-) \rightarrow L^\chi(\widetilde{\Gamma/B^+})$  given by

$$\begin{aligned} \eta: L^\chi(\widetilde{\Gamma/B^+}) &\longrightarrow Hol(N^-) & \zeta: Hol(N^-) &\longrightarrow L^\chi(\widetilde{\Gamma/B^+}) \\ f &\longrightarrow G_f := \eta(f) & G &\longrightarrow f_G := \zeta(G) \end{aligned}$$

where  $G_f$  and  $f_G$  are the morphisms defined as follows

$$G_f: N^- \xrightarrow{i} N^- \times B^+ \xrightarrow{\kappa} N^- B^+ \xrightarrow{f} \mathbb{C}^{1|1}$$

and

$$f_G: N^- B^+ \xrightarrow{\kappa^{-1}} N^- \times B^+ \xrightarrow{G \times \chi} \mathbb{C}^{1|1} \times GL(\mathbb{C}^{1|1}) \longrightarrow \mathbb{C}^{1|1}$$

$\eta$  and  $\zeta$  gives the desired isomorphism. ■

**Remark 3.11.** The identification  $L^\chi(\widetilde{\Gamma/B^+}) \simeq Hol(N^-)$  is automatically an isomorphism of Frechét superspaces. Indeed, (see Sec. 2.5) the topology on  $L^\chi$  is exactly the one induced by the local trivializations.

Precisely, recall that  $Hol(N^-) = \text{Hom}_{\mathfrak{U}(\mathfrak{n}^-)}(\mathfrak{U}(\mathfrak{n}^-), Hol(\widetilde{N^-}))$ . The superalgebra of global sections  $\text{Hom}_{\mathfrak{U}(\mathfrak{n}^-)}(\mathfrak{U}(\mathfrak{n}^-), Hol(N^-))$  is globally split:

$$Hol(N^-) \simeq Hol(\widetilde{N^-}) \otimes \Lambda^n$$

and we define the family of seminorms

$$q_{D,K}(f) = \sup_{x \in K} |\widetilde{D}f|$$

where  $K$  is a compact subset of  $N^-$ ,  $D$  is a differential operator on  $N^-$ .

To ease the notation we shall also write  $L^\chi(\Gamma)$  in place of  $L^\chi(\widetilde{\Gamma/B^+})$ .

We now want to study in detail the action of  $A_r$ , the ordinary torus in  $G_r$ , on the polynomials  $\mathcal{P}$  in  $Hol(N^-)$  and the corresponding superalgebra



$\mathcal{P}^\sim$  in  $F = L^\chi(\Gamma)$ . Let  $t_\alpha$  denote the global exponential coordinates on  $N^-$  (see Sec. 3.2).  $t_\alpha$  will be even or odd depending on whether  $\alpha$  is even or odd.

By a classical result, if  $N_0$  is a connected nilpotent Lie group,  $\mathcal{U}(\mathfrak{n}_0)$  preserves the polynomials  $\mathcal{P}(N_0)$  (in the even indeterminates  $t_\alpha$ ). We thus have the natural identifications:

$$\mathcal{P}(N^-) = \mathcal{P}(N_0^-) \otimes \wedge(\mathfrak{n}_1^-) = \underline{\text{Hom}}_{\mathfrak{n}_0}(\mathcal{U}(\mathfrak{n}^-), \mathcal{P}(N_0^-)) \subset \underline{\text{Hom}}_{\mathfrak{n}_0}(\mathcal{U}(\mathfrak{n}^-), C^\infty(N_0^-))$$

Notice that  $\mathcal{P}(N^-)$  are the polynomials in the (even and odd) indeterminates  $t_\alpha$ .

Proposition 3.10 allows us to obtain immediately the following corollary (we shall also see it later as a consequence of Lemma 3.16 in the next section).

**Proposition 3.12.**  *$\mathcal{P}$  is dense in  $\text{Hol}(N^-)$  and  $\mathcal{P}^\sim$  is dense in  $F$ .*

*Proof.* In view of the definition of the topology on  $F$ , it is enough to prove that  $\mathcal{P}$  is dense in  $\text{Hol}(N^-)$ . The proof goes as follows.  $\widetilde{N^-}$  is analytically isomorphic to  $\mathbb{C}^{m|n}$  via the exponential morphism (see Sec. 3.2), for suitable  $m|n$ . The algebra of polynomials  $\text{Pol}(\mathbb{C}^m)$  is dense in the algebra  $\text{Hol}(\mathbb{C}^m) \simeq \text{Hol}(\widetilde{N^-})$  of entire functions over  $\mathbb{C}^m$  with respect to the family of seminorms

$$q_K = \sup_{x \in K} |\cdot|$$

where  $K$  is a compact subset of  $\mathbb{C}^m$ . This comes from the theory in the ordinary setting.

Consider the super coordinate system  $t_\alpha$  on  $N^-$  defining the polynomial supersheaf and denote such coordinates more specifically with

$$(t_\alpha) = (z^u, \xi^v)$$

For any compact  $K$  and differential operator  $D = \sum_J a^J(z, \xi) \partial_{\xi^J}$ , a basis of seminorms is given by

$$q_{K,D}(f) := \sup_{x \in K} \left| a^J(z, \xi) \frac{\partial^{|J|} f}{\partial \xi^J} \right|$$

where we have adopted a multi-index notation. A generic element  $f$  in  $\text{Hol}(N^-)$  can be written as

$$\sum_P f_P(z^1, \dots, z^n) \xi^P$$

Fix now  $f$  and a seminorm  $q_{K,D}$ . For each  $\epsilon > 0$ , we have to find a super polynomial  $p$  such that

$$q_{K,D}(f - p) \leq \epsilon$$

Since

$$q_{K,D}(f - p) \leq M \sup_{x \in C} \sum_J |f_J - p_J|$$

where  $M$  is a positive constant, we are thus reduced to the classical case, and we are done. ■

### 3.5 The action of the maximal torus on the polynomials on the big cell

In this section we introduce two natural actions  $c$  and  $l$  of the ordinary Lie group  $A_r$  on the big cell  $\Gamma = N^- A N^+$ , together with the actions  $i$  and  $\ell$  they induce on the Frechét superspace  $L^\times(\Gamma)$ . We notice for future reference that both actions coincide on the quotient  $\Gamma/B^+$ .

The key step in the definition of the two actions is the isomorphism (see Prop. 3.6)

$$\kappa: N^- \times B^+ \xrightarrow{\cong} \Gamma$$

Let us start with the action  $c$  related to the conjugation. Since  $A_r$  acts on  $N^-$  by conjugation (see Prop. 3.9), we have a global action of  $A_r$  on  $\Gamma$  defined as:

$$c: A_r \times \Gamma \xrightarrow{1_{A_r} \times \kappa^{-1}} A_r \times (N^- \times B^+) \xrightarrow{\text{conj} \times 1_{B^+}} N^- \times B^+ \xrightarrow{\kappa} \Gamma \quad (23)$$

which in the functor of points notation reads (forgetting the identification  $\kappa$ ),

$$a \cdot (n^- b^+) = (a n^- a^{-1}) b^+, \quad a \in |A_r|, n^- \in N^-(T), b^+ \in B^+(T).$$

Since  $A_r$  also acts on  $B^+$  by left translation  $l'$ , we can define the left action of  $A_r$  on  $\Gamma$  as

$$l_a = \kappa \circ (\text{conj}_a \times l'_a) \circ \kappa^{-1} \quad (24)$$

or, in the functor of points notation,

$$a \cdot (n^- b^+) = (a n^- a^{-1}) a \cdot b^+.$$

Both actions commute with right translations by  $B^+$  and hence define representations of  $A_r$  on  $L^\chi(\Gamma)$

$$i, \ell: A_r \times L^\chi(\Gamma) \longrightarrow L^\chi(\Gamma)$$

where

$$i_a(f) = c_{a^{-1}}^*(f) \qquad \ell_a(f) = l_{a^{-1}}^*(f)$$

for all  $a \in \widetilde{A_r}$ , and all  $f \in L^\chi(\Gamma)$ .

These representations are most easily written in the functor of points notation as

$$\begin{aligned} i_a(f)(n^-b^+) &= f((a^{-1}n^-a)b^+) \\ \ell_a(f)(n^-b^+) &= f((a^{-1}n^-a)a^{-1}b^+) \end{aligned}$$

The above formulas further simplify using the identification

$$L^\chi(\Gamma) \simeq \text{Hol}(N^-),$$

we leave the details to the reader.

**Lemma 3.13.** *Let the notation be as above. Then*

1.  $\ell_a f = \chi(a)(i_a f)$
2.  $i_a t_\alpha = \chi_\alpha(a) t_\alpha \quad \forall a \in |A_r|$

*Proof.* (1) follows immediately from (24). For (2) let  $n = \exp(\sum_{\beta \in P} y_\beta X_{-\beta})$  be an element in  $N^-$ , then the result comes from the following formal calculation (see Sec. 3.2):

$$\begin{aligned} t_\alpha(a^{-1}na) &= t_\alpha\left(\exp\left(\sum_{\beta \in P} y_\beta \text{Ad}(a)X_{-\beta}\right)\right) = t_\alpha\left(\exp\left(\sum_{\beta \in P} y_\beta \chi_\beta(a)X_{-\beta}\right)\right) \\ &= \chi_\alpha(a) t_\alpha(n), \end{aligned}$$

where  $a \in |A_r|$ ,  $y_\beta \in \mathbb{C}$  and the  $t_\alpha$  are the polynomial coordinates on  $N^-$  (see Sec. 3.4). ■

**Proposition 3.14.** *Let  $\mathcal{P}$  be the polynomial superalgebra generated by the  $t_\alpha$  in  $\text{Hol}(N^-)$  and let  $\mathcal{P}^\sim$  be the corresponding submodule in  $F$ .  $A_r$  acts on  $\mathcal{P}^\sim$  and we have that:*

$$a \cdot \widetilde{t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}}} = \chi_{\lambda + \sum r_{\alpha_i} \alpha_i}(a) \widetilde{t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}}}$$

Hence  $\mathcal{P}^\sim$  decomposes into the sum of eigenspaces  $\mathcal{P}_d^\sim$  for the action of  $A_r$ , where  $d$  ranges in  $D^+$  the semigroup in  $\mathfrak{h}^*$  generated by the positive roots:

$$\mathcal{P}^\sim = \bigoplus_{d \in D^+} \mathcal{P}_d^\sim, \quad \mathcal{P}_d^\sim = \bigoplus_{\sum r_{\alpha_i} \alpha_i = d} \mathbb{C} \cdot \widetilde{t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}}}$$

A similar decomposition holds also for  $\mathcal{P}$ .

*Proof.* Let us do this just for  $t_\alpha^r$ , the general calculation being the same. By (1) and (2) of the previous lemma we have:

$$\begin{aligned} a \cdot \widetilde{t_\alpha^r} &= (a \cdot t_\alpha^r)(n) = t_\alpha^r(a^{-1}n) = t_\alpha^r(a^{-1}na)\chi_\lambda(a) = \\ &= t_\alpha^r(n)(\chi_\alpha(a))^r \chi_\lambda(a) = t_\alpha^r(n)\chi_{\lambda+r\alpha}(a). \end{aligned}$$

■

**Corollary 3.15.** *The maximal torus  $A_r$  acts on the Frechét superspace  $L^\times(\Gamma)$  and we have:*

1.  $F(\tau) \neq 0$  if and only if  $\tau = \chi_{-\lambda+d}$  for some  $d = \sum_{m_\alpha \in \mathbb{Z}_{\geq 0}, \alpha \in P} m_\alpha \alpha$ .
- 2.

$$F(\chi_{\lambda+d}) = \mathcal{P}_{\lambda+d}^\sim$$

and

$$\dim(F(\chi_{\lambda+d})) = \# \left\{ r = (r_\alpha) \mid \sum_{r_\alpha \in \mathbb{Z}_{\geq 0}, \alpha \in P} r_\alpha \alpha = d \right\}$$

*Proof.* (1) and (2) are consequences of Lemma 2.41. The computation of the dimension is straightforward. ■

We now want to prove the fact that the spectrum of  $A_r$  goes unchanged if we change the open set we are considering in a suitable way. We shall first prove a general lemma.

Let  $T$  be an ordinary compact torus acting on a finite dimensional complex vector superspace  $V$ . We may assume without loss of generality that  $V \cong \mathbb{C}^{m|n}$ . By a classical result we have that: the action of  $T$  on  $V$  is via characters  $\tau_i$ 's and reads as follows:

$$T \times V \longrightarrow V$$

$$t, (v_1, \dots, v_m, \nu_1, \dots, \nu_n) \longmapsto (\tau_1(t)v_1, \dots, \tau_m(t)v_m, \tau_{m+1}(t)\nu_1, \dots, \tau_{m+n}(t)\nu_n)$$

where the  $v_i$ 's and  $\nu_j$ 's are complex numbers. We can easily transport this action to the space of polynomial functions  $\text{Pol}(V)$  on  $V$  and obtain the following action:

$$t \cdot \sum a_{IJ} z^I \xi^J = \sum a_{IJ} \tau^I(t)^{-1} \tau^J(t)^{-1} z^I \xi^J$$

using the multiindex notation  $I = (i_1, \dots, i_n)$  with possibly repeated indeces,  $J = (j_1, \dots, j_n)$  with no repeated indeces.

$T$  has also a natural action on the holomorphic sections of the structural sheaf on  $V$ ,  $\mathcal{O}_V(U)$ , where  $U$  is a  $T$  invariant open set in  $V$ . If  $g \in \mathcal{O}_V(U)$ , we know we can view such  $g$  as  $g : U \longrightarrow \mathbb{C}^{1|1}$ . If  $\rho_t(u) := t \cdot u$  is the action of  $T$  on  $U$ , we define:

$$t \cdot g = g \circ \rho_{t^{-1}}$$

Notice that such action agrees with the previously defined action on the polynomials.

We define  $\text{Pol}(\tau)$  the space of polynomials transforming according to the character  $\tau$ , that is:

$$\text{Pol}(\tau) = \{p \in \text{Pol}(V) \mid t \cdot p = \tau(t)p\}$$

**Lemma 3.16.** *Let  $T$  be an ordinary compact torus acting on a finite dimensional complex vector superspace  $V$ . For any character  $\tau$  of  $T$ , we assume that  $\dim(\text{Pol}(\tau)) < \infty$ . Then any open connected subset  $U$  of  $V$  which is  $T$ -invariant and contains the origin, is such that  $\text{Pol}(U)$  is dense in  $\mathcal{O}_V(U)$ .*

*Proof.* We may assume that  $V = \mathbb{C}^{m|n}$  with  $T$ -action

$$t, (z_1, \dots, \xi_{m+n}) \longmapsto (f_1(t)z_1, \dots, f_{m+n}(t)\xi_{m+n})$$

where the  $f_j$  are characters of  $T$ . Let  $U$  be an open connected subset of  $\mathbb{C}^{m|n}$  containing the origin and stable under  $T$ . The action of  $T$  induces an

action on  $\mathcal{O}_V(U)$ . It is enough to prove that the closure of  $\text{Pol}(\mathbb{C}^{m|n})$  contains  $\mathcal{O}_V(U)^\infty$  the smooth vectors in  $\mathcal{O}_V(U)$  with respect to the  $T$  action, since we know such space is dense in  $\mathcal{O}_V(U)$ . Since the Fourier series of any  $g$  in  $\mathcal{O}_V(U)$  converges to  $g$  (see Prop. 2.39 (5) and (6)), it is enough to show that any eigenfunction of  $T$  in  $\mathcal{O}_V(U)$  is a polynomial. Suppose  $g \neq 0$  is in  $\mathcal{O}_V(U)$  such that  $t^{-1} \cdot g = f(t)g$  for all  $t \in T$  and  $u \in U$ ,  $f$  being a character of  $T$ . Since  $0 \in U$  we can expand  $g$  as a power series  $g(u) = \sum_r c_r u^r$  in a polydisk, where  $u$  comprehends even and odd coordinates and we are using the multiindex notation. Notice that the action of  $T$  preserves the polydisks, and we have

$$(t^{-1} \cdot g)(u) = g(tu) = \sum_r c_r (tu)^r = \sum_r c_r f^r(t) u^r = f(t)g = \sum_r c_r f(t) u^r$$

Then  $c_r f^r = c_r f$  whenever  $c_r \neq 0$ , because  $t^{-1} \cdot g = f(t)g$ . So only the  $r$  with  $f = f^r$  appear in the expansion of  $g$ . We claim that there are only finitely many such  $r$ ; once this claim is proven we are done, because  $g$  is a linear combination of the monomials  $u^r$  with  $f^r = f$ , hence  $g$  is a polynomial. To prove the claim, note that all such  $u^r$  are eigenfunctions for  $T$  for the eigencharacter  $f$ , and by assumption, there are only finitely many of these. ■

We want to apply the previous lemma in a case that is of interest to us.

Define now  $\Gamma_1 = G_r B^+ / B^+$  and  $\Gamma_2 = (\Gamma \cap G_r B^+)^0 / B^+$  (the suffix “0” denotes the connected component of the identity). These are open sets in  $G/B^+$  which are invariant under the  $A_r$  action. R. e’  $G_r B^+$  e non  $G_0 B^+$

**Corollary 3.17.** *Let the notation be as above.*

1.  $\overline{\mathcal{P}} = \mathcal{O}(N^-)$ , where the nilpotent supergroup  $N^-$  is interpreted as a vector superspace via the identification  $\mathfrak{n}^- \cong N^-$  via the exponential morphism.
2.  $\overline{\mathcal{P}^\sim} = L^\chi(\Gamma)$ ,  $\Gamma = N^- A N^+$  the big cell in  $G$ .
3.  $\overline{\mathcal{P}^\sim}(\Gamma_2) = L^\chi(\Gamma_2)$ .

*Proof.* (1) We apply Lemma 3.16. The torus  $A_r$  acts on  $N^-$  through the action  $c$  given by (23). The condition  $\dim \text{Pol}(\tau) < \infty$  is checked with a calculation completely similar to that of Corollary 3.15.

(2) is a consequence of the topological isomorphism (22).

(3) follows again from Lemma 3.16. ■

Define now  $F = L^\chi(\Gamma)$ ,  $F^1 = L^\chi(\Gamma_1)$ ,  $F^2 = L^\chi(\Gamma_2)$ . Notice that on  $F$  and  $F^2$  we do not have any  $G$  or  $G_r$  action, only  $F^1$  is a  $G_r$  module in a natural way.

**Corollary 3.18.** *Let the notation be as above.*

1. *The restriction morphism  $F^1 \longrightarrow F^2$  is a continuous injection.*
2. *Under the restriction,  $F^1(\tau) \subset F^2(\tau)$  for characters  $\tau$  of  $A_r$ .*
3.  *$F^2(\tau) = F(\tau)|_{\Gamma_2}$ .*

*Proof.* (1) and (2) are clear, (1) because  $\Gamma_2$  is open in  $\Gamma_1$  and of the analytic continuation principle, which holds also in the supersetting, while (2) is a simple check. Now we go to (3). The space of polynomials  $\text{Pol}(\Gamma)$  on  $\Gamma$  is dense in  $F$  and by Corollary 3.15 we have  $F^0 = \text{Pol}(\Gamma)$ .  $\text{Pol}(\Gamma_2) = \text{Pol}(\Gamma)|_{\Gamma_2}$  is dense in  $F^2$  by the previous corollary. Since  $\text{Pol}(\Gamma_2)$  is dense in  $F^2$ , we have that the restriction of  $F(\tau)$  to  $\Gamma_2$  is dense in  $F^2(\tau)$  and since  $F(\tau)$  is finite dimensional we have  $F(\tau)|_{\Gamma_2} = F^2(\tau)$ . ■

### 3.6 The action of $\mathcal{U}(\mathfrak{g})$ on $L^\chi(\Gamma)$

We start by defining the natural action of  $\mathcal{U}(\mathfrak{g})$  on the holomorphic functions on any neighbourhood  $W$  of the identity of the supergroup  $G$ .

**Definition 3.19.** Let  $W \subset G$  be an open neighbourhood of the identity  $1_G$  in  $G$ . There are two well defined actions of  $\mathfrak{g}$ , hence of  $\mathcal{U}(\mathfrak{g})$ , on  $\text{Hol}(W)$  that read as follows:

$$\begin{aligned}\ell(X)f &= (-X \otimes 1)\mu^*(f), & X \in \mathfrak{g} \\ \partial(X)f &= (1 \otimes X)\mu^*(f)\end{aligned}$$

**Proposition 3.20.** *Let  $U$  be open in  $\widetilde{G/B}$ . Then  $\ell$  and  $\partial$  are well defined actions on  $L^\chi(U)$  and they commute with each other.*

*Proof.* Immediate. ■

We want to show that the natural action of  $\mathcal{U}(\mathfrak{g})$  on  $L^\chi(N^-B)$  preserves  $\mathcal{P}^\sim(N_0^-)$ . For this we need some preliminary notation. Since  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}^+$ , if we fix bases of  $\mathfrak{n}^-$  and  $\mathfrak{b}^+$ , by the PBW theorem any  $X \in \mathcal{U}(\mathfrak{g})$  can be written (with obvious meaning of the symbols) as:

$$X = \sum_{I,J} c_{IJ}(X) B_I N_J \tag{25}$$

**Lemma 3.21.** *Let  $\phi \in \mathcal{O}_G(N^-B)$ .  $\phi$  is in  $L^\chi(N^-B)$  if and only if*

$$\phi(X)(nb) = \chi_0(b)^{-1} \sum_{IJ} c_{IJ}(b.X) \lambda(\overline{B}_I) \phi(N_J)(n), \quad X \in \mathcal{U}(\mathfrak{g}),$$

where  $b.X$  is the adjoint action of  $b \in B_0$  on  $\mathcal{U}(\mathfrak{g})$  and as usual  $\overline{B}_I$  denotes the antipode of  $B_I$  in the Hopf superalgebra  $\mathcal{U}(\mathfrak{g})$ .

*Proof.* By the very definition we have  $\phi \in L^\chi(N^-B)$  if

1.  $r_b^* \phi = \chi_0(b)^{-1} \phi$ ,  $b \in B_0$ .
2.  $D_Y^L(\phi) = \lambda(\overline{Y}) \phi$ ,  $\lambda|_{\mathfrak{g}_0} = d\chi_0$ .

The result comes with a calculation. ■

Notice that once the lemma is established, if  $p$  is a polynomial in  $\mathcal{P}(N^-)$  we can define  $p^\sim \in L^\chi(N^-B)$  as:

$$p^\sim(X)(nb) = \chi_0(b)^{-1} \sum_{IJ} c_{IJ}(b.X) \lambda(\overline{B}_I) p(N_J)(n)$$

Vice-versa we can recover  $p$  from  $p^\sim$  restricting to  $N^-$ . In the language of SHCP this amounts to two restrictions: we impose  $b = 1$  and  $X \in \mathcal{U}(\mathfrak{n}^-)$ .

**Proposition 3.22.** *The actions  $\ell$  and  $\partial$  of  $\mathcal{U}(\mathfrak{g})$  on  $L^\chi(U)$ ,  $p^{-1}(U) \subset \Gamma$  leave  $\mathcal{P}_\chi^\sim$  invariant.*

*Proof.* We start with the action  $\ell$  (for  $\partial$  the reasoning is the same, the calculations easier). We need to show that, given  $Z \in \mathcal{U}(\mathfrak{g})$  and  $X \in \mathcal{U}(\mathfrak{n}^-)$ ,  $(D_Z^R p^\sim|_{N^-})(X) \in \mathcal{P}(N_0^-)$ . We have (see [8] Sec. 7.4):

$$(D_Z^R p^\sim)(X)(g) = (-1)^{|Z||p|} [p^\sim((g^{-1}.Z)X)](g)$$

Hence if  $n \in N^-$ , we have:

$$\begin{aligned} (D_Z^R p^\sim)(X)(n) &= (-1)^{|Z||p|} [p^\sim((n^{-1}.Z)X)](n) \\ &= (-1)^{|Z||p|} \sum_{IJ} c_{IJ}((n^{-1}.Z)X) [\lambda(\overline{B}_I) p^\sim(N_J)](n) \end{aligned}$$

where  $B_I$  and  $N_J$  are obtained as in (25) applied to  $(n^{-1}.Z)X$ . The last equality is true by Lemma 3.21. ■



We now want to establish a fundamental pairing between a certain Verma module and the space of polynomials inside  $L^\chi(\Gamma)$ . We start by recalling the notion of pairing between  $\mathfrak{g}$ -modules.

**Definition 3.23.** Let  $M_1, M_2$  be two modules for  $\mathfrak{g}$ . By a  $\mathfrak{g}$ -pairing between them we mean a bilinear form  $\langle \cdot, \cdot \rangle$  on  $M_1 \times M_2$  with the property that:

$$\langle Xm_1, m_2 \rangle = \langle m_1, -(-1)^{|m_1||X|}Xm_2 \rangle, \quad m_i \in M_i, X \in \mathfrak{g}.$$

Since the  $M_i$  are modules for  $\mathcal{U}(\mathfrak{g})$  this implies that

$$\langle X_1 \dots X_r m_1, m_2 \rangle = \langle m_1, (-1)^{r+|m_1|(|X_1|+\dots+|X_r|)+l_{\text{odd}}(w)} X_r \dots X_1 m_2 \rangle$$

$$m_i \in M_i, X_j \in \mathfrak{g}.$$

where  $l_{\text{odd}}(w)$  is the (minimum) number of odd transpositions appearing in the permutation  $w : (1, \dots, r) \mapsto (r, \dots, 1)$ . The map  $X \mapsto -X$  of  $\mathfrak{g}$  is an involutive anti-automorphism of  $\mathfrak{g}$ . It extends uniquely to an involutive anti-automorphism  $u \mapsto u^T$  of  $\mathcal{U}(\mathfrak{g})$ . The  $\mathfrak{g}$ -pairing requirement is equivalent to

$$\langle um_1, m_2 \rangle = \langle m_1, (-1)^{|u||m_1|}u^T m_2 \rangle, \quad m_i \in M_i, u \in \mathcal{U}(\mathfrak{g}).$$

We refer to this as a  $\mathcal{U}(\mathfrak{g})$ -pairing also. The pairing is said to be *non-singular* if  $\langle m_1, m_2 \rangle = 0$  for all  $m_2$  (resp. for all  $m_1$ ) implies that  $m_1 = 0$  (resp.  $m_2 = 0$ ).

**Proposition 3.24.** Let  $u$  and  $v$  in  $\mathcal{U}(\mathfrak{g})$ ,  $f \in \mathcal{P}^\sim$ .

1.

$$(\partial(u^T)f)(1_G) = (\ell(u)f)(1_G)$$

2.

$$\partial(u)\ell(v)(f)(1_G) = (-1)^{|u||v|}\ell(v)\partial(u)(f)(1_G)$$

where  $1_G$  denotes the identity element in  $G$ .

*Proof.* (1). It is enough to prove for  $u = X$  and  $v = Y$  both in  $\mathfrak{g}$ . We can rewrite our equality as:

$$(\epsilon \otimes 1)(1 \otimes -X)\mu^*(f) = (1 \otimes \epsilon)(-X \otimes 1)\mu^*(f)$$

where  $\epsilon$  is the counit morphism:  $\epsilon(f) = f(1_G) \quad \forall f \in \mathcal{P}^\sim$ . We have:

$$(\epsilon \otimes 1)(1 \otimes X)\mu^*(f) = (1 \otimes X)(\epsilon \otimes 1)\mu^*(f) = X(f)$$

since  $(\epsilon \otimes 1)\mu^*(f) = f$ . On the other hand:

$$(1 \otimes \epsilon)(X \otimes 1)\mu^*(f) = (X \otimes 1)(1 \otimes \epsilon)\mu^*(f) = X(f).$$

(2). Again it is enough to prove for  $u = X$  and  $v = Y$  both in  $\mathfrak{g}$ .

$$\begin{aligned} \partial(X)\ell(Y)(f)(1_G) &= (\epsilon \otimes 1)(1 \otimes X)\mu^*(-Y \otimes 1)\mu^*(f) = \\ &= (1 \otimes X)(\epsilon \otimes 1)\mu^*(-Y \otimes 1)\mu^*(f) = \\ &= (-1)^{|X||Y|}(-Y \otimes X)\mu^*(f)(1_G). \end{aligned}$$

because  $(\epsilon \otimes 1)\mu^*(f) = f$ . Similarly

$$\begin{aligned} \ell(Y)\partial(X)(f)(1_G) &= (1 \otimes \epsilon)(-Y \otimes 1)\mu^*(1 \otimes X)\mu^*(f) = \\ &= (-Y \otimes 1)(1 \otimes \epsilon)\mu^*(1 \otimes X)\mu^*(f) = \\ &= (-Y \otimes X)\mu^*(f)(1_G). \end{aligned}$$

■

**Lemma 3.25.** *Let  $\lambda \in \mathfrak{h}^*$ , and let*

$$\mathcal{M}_\lambda := \sum_{\alpha > 0} \mathcal{U}(\mathfrak{g})\mathfrak{g}_\alpha + \sum_{H \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(H + \lambda(H))$$

*then  $\mathcal{M}_\lambda$  is a left ideal and*

$$\mathcal{U}(\mathfrak{g}) = \mathcal{M}_\lambda \oplus \mathcal{U}(\mathfrak{n}^-)$$

*Proof.* For the ordinary setting this is Lemma 4.6.6 in [42]. As for the super-setting it is the same. ■

**Theorem 3.26.** *There is a non-singular  $\mathcal{U}(\mathfrak{g})$ -pairing between  $\mathcal{P}^\sim$  and the Verma module  $V_\lambda$ . Moreover every non-zero submodule of  $\mathcal{P}^\sim$  contains the element  $1^\sim$  corresponding to the constant function  $1 \in \mathcal{P}$ . In particular, the submodule  $\mathcal{I}^\sim$  of  $\mathcal{P}^\sim$  generated by  $1^\sim$  is irreducible and is the unique irreducible submodule of  $\mathcal{P}^\sim$ . Finally,  $\mathcal{I}^\sim$  is the unique irreducible module of lowest weight  $-\lambda$ .*

*Proof.* The proof is the same as for the ordinary setting, let us sketch it. We first define:

$$\langle, \rangle: \mathcal{P}^\sim \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}, \quad \langle f, u \rangle := (-1)^{|u||f|} (\partial(u)f)(1_G)$$

In order for  $\langle, \rangle$  to be a  $\mathfrak{g}$ -pairing, we need to verify:

$$\langle \ell(c)f, u \rangle = \langle f, (-1)^{|f||c|} c^T u \rangle, \quad c, u \in \mathcal{U}(\mathfrak{g}), f \in \mathcal{P}^\sim$$

where  $(\cdot)^T$  denotes the antiautomorphism of  $\mathcal{U}(\mathfrak{g})$  induced by  $X \mapsto -X$  with  $X \in \mathfrak{g}$ . We have by the previous proposition:

$$\begin{aligned} \langle \ell(c)f, u \rangle &= (-1)^{|u|(|c|+|f|)} (\partial(u)\ell(c)f)(1_G) = (-1)^{|u||f|} (\ell(c)\partial(u)f)(1_G) = \\ &= (-1)^{|u||f|} (\partial(c^T)\partial(u)f)(1_G) = \\ &= (-1)^{|u||f|} (\partial(c^T u)f)(1_G) = \langle f, (-1)^{|f||c|} c^T u \rangle \end{aligned}$$

By Lemma 3.25, in order to prove that the bilinear map  $\langle, \rangle$  descends to a  $\mathfrak{g}$ -pairing between  $\mathcal{P}^\sim$  and  $V_\lambda$  we need to prove that

$$u \in \mathcal{M}_\lambda \iff (\partial(u)f)(1_G) = 0$$

For sufficiency, we notice that  $\langle f, X_\alpha \rangle = \partial(X_\alpha)(f)(1_G) = D_{X_\alpha}^L(f) = \lambda(-X_\alpha) = 0$  by (21). Again by (21), we have that  $\langle f, H \rangle = D_H^L(f) = -\lambda(H)f(1_G)$ . For necessity, suppose that  $(\partial(u)f)(1_G) = 0$  for each  $f \in \mathcal{P}^\sim$ . By Lemma 3.25 it is enough to notice that for each  $X \in \mathcal{U}(\mathfrak{n}^-)$  there exists a polynomial  $p \in \mathcal{P}$  such that  $D_X^L p^\sim(1_G) \neq 0$ . So we have obtained a nonsingular pairing

$$\mathcal{P}_\lambda^\sim \subset V_{-\lambda}^*$$

They are both weight spaces, for each weight the corresponding weight spaces having the same dimension (See Cor. 3.15), hence they are isomorphic.

More explicitly, the functions  $(t^r)^\sim = (t_{\alpha_1}^{r_1} \dots t_{\alpha_m}^{r_m})^\sim$ , corresponding to the coordinate polynomials  $t^r$  on  $N^-$ , are weight vectors for the action of  $\mathfrak{h}$  for the weight  $r - \lambda$ . Hence  $\mathcal{P}^\sim$  is a weight module with the multiplicities defined in Sec. 3.5. We shall prove that every non-zero  $\ell$ -invariant subspace  $W$  of  $\mathcal{P}^\sim$  contains the vector  $1^\sim$  defined by the constant function 1 on  $N$ . Now  $W$  is a sum of weight spaces and if it does not contain  $1^\sim$ , then  $W$  is contained in the sum of all weight spaces corresponding to the weights  $\lambda - r$  where

$r = (r_i)$  with some  $r_i > 0$ . Now  $\langle Hm_1, m_2 \rangle = (-1)^{|m_1||H|} \langle m_1, Hm_2 \rangle$ , for all  $H \in \mathfrak{h}$ ,  $m_1 \in \mathcal{P}^\sim$ ,  $m_2 \in V_\lambda$ . This shows that the weight space of  $\mathcal{P}^\sim$  for the weight  $\theta$  is orthogonal to the weight space of  $V_\lambda$  for the weight  $\phi$  unless  $\theta = \phi$ . Let  $v$  be a non-zero vector of highest weight  $\lambda$  in  $V_\lambda$ . Since  $W$  is contained in the span of weights other than  $-\lambda$ , we have  $\langle W, v \rangle = 0$ . Hence, for all  $g \in \mathcal{U}(\mathfrak{g})$ ,  $w \in W$  we have  $\langle \ell(g)w, v \rangle = 0$ . So  $\langle w, g^T v \rangle = 0$  for all  $g \in \mathcal{U}(\mathfrak{g})$ . But  $v$  is cyclic for  $V_\lambda$  and so we have  $\langle w, V_\lambda \rangle = 0$  for all  $w \in W$ . This means that  $W = 0$ , contradicting the hypothesis that  $W \neq 0$ . Thus every non-zero submodule of  $\mathcal{P}^\sim$  contains the submodule  $\mathcal{I}^\sim$  generated by  $1^\sim$ . This submodule is then the unique irreducible submodule of  $\mathcal{P}^\sim$ . The weights of  $\mathcal{I}^\sim$  are of the form  $-\lambda + d$  where  $d$  is a positive integral linear combination of the simple roots, and  $1^\sim$  has weight  $-\lambda$ . It is then clear that  $1^\sim$  is the lowest weight of  $\mathcal{I}^\sim$ . This fact, together with its irreducibility, characterizes it uniquely. ■

### 3.7 Harish-Chandra decomposition

In this section we want to briefly recall the notion of *admissible system* for  $\mathfrak{g}$  and to show that it leads to the *Harish-Chandra decomposition*, (for more details see [10] and [14]). The existence of admissible systems is classically linked to a natural invariant complex structure on the symmetric space  $G_r/K_r$ , where  $G_r$  is a semisimple real Lie group and  $K_r$  its maximal compact ordinary subgroup. As we shall see, here the situation is very similar.

Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the decomposition of  $\mathfrak{g}$  with respect to the Cartan automorphism uniquely associated with the real form  $\mathfrak{g}_r = \mathfrak{k}_r \oplus \mathfrak{p}_r$ , which also decomposes through a real Cartan automorphism,  $\mathfrak{k}$  and  $\mathfrak{p}$  being the complexifications of  $\mathfrak{k}_r$  and  $\mathfrak{p}_r$  respectively.

The positive roots whose root spaces are in  $\mathfrak{k}$  are called *compact* and their set denoted with  $P_k$ , the positive roots whose root spaces are in  $\mathfrak{p}$  are called *non compact* and their set denoted with  $P_n = P_{n,0} \cup P_{n,1}$ , where the indices 0 and 1 indicate the set of even and odd roots respectively.

We say that a positive system  $P$  is *admissible* if  $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$  as  $\mathfrak{k}$ -module,  $\mathfrak{p}^\pm = \sum_{\alpha \in \pm P_n} \mathfrak{g}_\alpha$  and  $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$ . We notice that, while in the ordinary setting the requirement for  $\mathfrak{p}^\pm$  is to be abelian, for its super counterpart we only ask it to be a Lie subsuperalgebra of  $\mathfrak{g}$ .

We say that a root is *totally positive* if its root space is contained in a subspace of  $\mathfrak{p}^+$  stable under the adjoint action of  $\mathfrak{k}$ . Evidently the positive non compact roots in an admissible system are totally positive and admissible systems are characterized by this property.

We have the following important result (see [10] Theorem 3.3 and [14] Theorem 1.1).

**Theorem 3.27.** *1. Let  $P_0$  be an admissible system for  $\mathfrak{g}_0^{ss}$ , the semisimple part of  $\mathfrak{g}_0$ . Then  $\mathfrak{g}$  has an admissible system  $P$  containing  $P_0$ .*

*2. A necessary condition for the existence of totally positive roots and admissible systems is that the center  $\mathfrak{c}$  of  $\mathfrak{k}$  is non zero. When  $\mathfrak{c} \neq 0$ , there exactly  $2^n$  admissible positive system for  $\mathfrak{g}$  containing a given positive system for  $\mathfrak{k}$ , where  $n$  is the number of simple components of  $\mathfrak{g}_0^{ss}$ .*

In [10] there is a complete list of admissible systems for basic classical Lie superalgebras. From now on we fix a positive admissible system  $P$  for  $\mathfrak{g}$ .

Let  $P^+$  be the supergroup corresponding to the subalgebra  $\mathfrak{p}^+ = \sum_{\beta \in P} \mathfrak{g}_\beta$ . Similarly define  $P^-$ . Let  $K$  be the (ordinary) complex subgroup of  $G$  corresponding to the Lie superalgebra  $\mathfrak{k} = \mathfrak{k}_0$ .

**Proposition 3.28.** *The morphism  $\phi : P^- \times K \times P^+ \longrightarrow G$ , defined as  $(p^-, k, p^+) \mapsto p^- k p^+$  in the functor of points notation, is a complex analytic diffeomorphism of  $P^- \times K \times P^+$  onto an open set  $\Omega \subset G$ .*

*Proof.* By the admissible systems theory we have that  $\mathfrak{k} + \mathfrak{p}^+$  is a Lie subalgebra of  $\mathfrak{g}$  and consequently there is a Lie supergroup  $S$  corresponding to it. By the classical theory  $|S|$  is closed and  $|S| = |K||P^+|$ . We apply Lemma 2.18 with  $M = S$ ,  $A_1 = K$  and  $A_2 = P^+$  and we obtain that  $KP^+ := \alpha(K \times P^+)$  is an open full submanifold of  $S$ . Since the morphism  $\alpha$  is bijective on the topological space and the superdimensions of  $KP^+$  and  $S$  are the same, we have  $KP^+ = S$ . Furthermore since  $|K| \cap |P^+| = 1$  we have  $K \times P^+ \cong KP^+ \cong S$ . We now apply the Lemma 2.18 again, with  $M = G$ ,  $A_1 = P^-$ ,  $A_2 = KP^+$ . By the ordinary theory (see [26] pg. 389) we have that  $|P^-| \cap |K||P^+| = \{1\}$ , and by the study on the super admissible systems we know that  $\mathfrak{p}_1^- \cap (\mathfrak{k} + \mathfrak{p}_1^+) = (0)$   $\mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ = \mathfrak{g}$  we have an analytic diffeomorphism  $P^- \times KP^+$  with  $P^- KP^+$  which is the open supermanifold image of the morphism  $(p^-, k, p^+) \mapsto p^- k p^+$ . Hence  $P^- \times K \times P^+$  is diffeomorphic to  $\Omega := P^- KP^+$ , which is open in  $G$ . ■

**Proposition 3.29.** 1.  $G_r KP^+$  is open and full in  $P^- KP^+$ .

2.  $|G_r| \cap |K| |P^+| = |K_r|$ .

*Proof.* For the first statement, observe that  $\mathfrak{g}_r \oplus \mathfrak{k} \oplus \mathfrak{p}^+ = \mathfrak{g}$ , by Prop. 3.5, because  $\mathfrak{k} \oplus \mathfrak{p}^+ \supset \mathfrak{b}$ . Hence by Lemma 2.18 we have that  $G_r KP^+$  is open and full in  $G$  and since  $|G_r| |K| |P^+| \subset |P^-| |K| |P^+|$  (see [26] pg. 389), we have  $G_r KP^+$  is open in  $P^- KP^+$ . The second statement is topological, so it is true because of the ordinary theory. ■

We now turn to the construction of the complex structure of  $G_r/K_r$ .

**Proposition 3.30.** We have  $G_r/K_r \cong G_r KP^+/KP^+$  and  $G_r/K_r$  is an hermitian superspace in the sense of Def. 3.2.

*Proof.* As abstract groups we have:

$$G_r(T)K(T)P^+(T)/K(T)P^+(T) \cong G_r(T)/G_r(T) \cap K(T)P^+(T)$$

If we can show  $G_r(T) \cap K(T)P^+(T) = K_r(T)$  then we are done since we obtain the equality of the two given quotients by taking their sheafification.

In the SHCP notation it amounts to show that:

$$(G_{r,0}, \mathfrak{g}_r) \cap (K_0 P_0^+, \mathfrak{k} \oplus \mathfrak{p}^+) = (K_{r,0}, \mathfrak{k}_{r,0})$$

This translates into two statements: one on the ordinary Lie groups involved, and this is true by the ordinary theory, the other on the Lie superalgebras namely

$$\mathfrak{g}_r \cap (\mathfrak{k} \oplus \mathfrak{p}^+) = \mathfrak{k}_r \tag{26}$$

The conjugation  $\sigma$  on  $\mathfrak{g}$  whose fixed points are  $\mathfrak{g}_r$  sends  $\mathfrak{g}_\alpha$  in  $\mathfrak{g}_{-\alpha}$  (see 3.5), since the roots are imaginary. Hence

$$\sigma(\mathfrak{k} \oplus \mathfrak{p}^+) = \sigma\left(\sum_{\alpha \in P_k \cup -P_k} \mathfrak{g}_\alpha \oplus \sum_{\beta \in P_n} \mathfrak{g}_\beta\right) = \sum_{\alpha \in P_k \cup -P_k} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta \in P_n} \mathfrak{g}_{-\beta}$$

Since  $\mathfrak{g}_r \cap (\mathfrak{k} \oplus \mathfrak{p}^+)$  consists of the elements in  $\mathfrak{k} \oplus \mathfrak{p}^+$  fixed by  $\sigma$  we obtain (26).

We now notice that the SHCP statement, translated into the functor of points notation, gives that:

$$(G_r \cap KP^+)(T) = G_r(T) \cap K(T)P^+(T) = K_r(T).$$

We now come to the proof that  $G_r/K_r$  is hermitian. By the first part of this proof,  $G_r/K_r$  acquires a complex supermanifold structure and furthermore the multiplication by  $g \in |G_r|$  is a supermanifold morphism. By Prop. 3.3 we obtain the result.  $\blacksquare$

### 3.8 Harish-Chandra representations and their geometric realization

This section is the heart of our treatment and gives a geometric and global realization of the Harish-Chandra infinitesimal representations studied in [10].

**Definition 3.31.** Let the complex vector superspace  $V$  be a  $\mathfrak{g}$ -module via the representation  $\pi$ . We say that  $V$  is a  $(\mathfrak{g}_r, K_r)$ -module if there exists a representation  $\pi_{K_r}$  of  $K_r$  such that

1.  $\pi(\text{Ad}(k)X) = \pi_{K_r}(k)\pi(X)\pi_{K_r}(k)^{-1}$
2.  $V = \sum_{\tau} V(\tau)$  where the sum is algebraic and  $V(\tau)$  is the span of all the linear finite dimensional subspaces corresponding to the irreducible representation associated with the  $K_r$ -character  $\tau$ .

We say that  $V$  is a  $(\mathfrak{g}_r, \mathfrak{k}_r)$ -module if

$$V = \sum_{\theta \in \Theta} V(\theta)$$

where the sum is algebraic,  $\Theta$  denotes the set of equivalence classes of the finite dimensional irreducible representations of  $\mathfrak{k}$  and  $V(\theta)$  is the sum of all representation occurring in  $V$  and laying in one of such classes  $\theta \in \Theta$ .

We now turn to the most important definition for the present work.

**Definition 3.32.** We say that the  $(\mathfrak{g}_r, K_r)$ -module  $V$  is an *Harish-Chandra module* (or HC-module for short) if  $V(\tau)$  is finite dimensional. Similarly we can define also the notion of Harish-Chandra modules for  $(\mathfrak{g}_r, \mathfrak{k}_r)$ -modules.

We say that a vector is  *$K_r$ -finite* if it lies in a finite dimensional  $K_r$  stable subspace.

For the reader's convenience we include two results found in [10] regarding HC-modules, that we shall need in the sequel.

**Proposition 3.33.** *Let  $U$  be a highest weight  $\mathfrak{g}$ -module with highest weight  $\lambda$  and highest weight vector  $v$ . Then the following are equivalent:*

1.  $\dim(\mathcal{U}(\mathfrak{k})v) < \infty$ ;
2.  $U$  is a  $(\mathfrak{g}, \mathfrak{k})$ -module;
3.  $U$  is a HC-module.

*If these conditions are true, then  $\mathcal{U}(\mathfrak{k})v$  is an irreducible  $\mathfrak{k}$ -module.*

**Theorem 3.34.** *Let  $U^\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}^+ \oplus \mathfrak{k})} F$ , where  $F$  is the irreducible (finite dimensional) module for  $\mathfrak{k}$  of highest weight  $\lambda$ .*

1.  $U^\lambda$  is the universal HC-module of highest weight  $\lambda$ .
2.  $U^\lambda$  has a unique irreducible quotient, which is the unique (up to isomorphism) irreducible highest weight Harish-Chandra module with highest weight  $\lambda$ .
3. If  $(\lambda + \rho)(H_\gamma) \leq 0$  for all  $\gamma \in P_n$  and  $< 0$  for  $\gamma$  isotropic, then  $U^\lambda$  is irreducible.

We now want to study the action of  $G_r$  on a superspace of sections of the line bundle  $L^\chi$  over  $\widetilde{G/B}$ . Since  $\widetilde{G_r B^+}$  is open in  $\widetilde{G}$  (see Lemma 2.19), we can consider  $L^\chi(G_r B/B)$  and since  $G_r$  acts on the left on  $G_r B^+$  we have a well defined action of  $G_r$  on the Frechét superspace  $L^\chi(G_r B^+/B^+)$ :

$$\begin{cases} (g \cdot f) = l_{g^{-1}}^* f & g \in \widetilde{G_r} \\ X.f = D_X^R f & X \in \mathfrak{g}_\mathbb{C} \end{cases}$$

where, as usual,  $\overline{X}$  is the antipode of  $X \in \mathcal{U}(\mathfrak{g})$ .

In the functor of points notation this is expressed as:

$$(g \cdot f)(x) = f(g^{-1}x)$$

according to Proposition 2.36.

Next lemma is a simple generalization of Theorem 11, pg 312 in [44] and holds in a general setting.



**Lemma 3.35.** *Let  $G_r$  be a connected Lie supergroup,  $\mathfrak{g}_r = \text{Lie}(G_r)$  its Lie superalgebra and  $\mathfrak{g}$  its complexification. Let  $F$  be a Frechét representation of  $G_r$  on which  $G_r$  acts via  $\pi = (\pi_0, \rho)$ . If  $v$  is a weakly analytic vector for  $\widetilde{G}_r$ , then*

$$\overline{\mathcal{U}(\mathfrak{g})v} \subseteq F$$

*and  $\overline{\mathcal{U}(\mathfrak{g})v}$  is the smallest closed  $G_r$ -invariant subspace of  $F$  containing  $v$ .*

*Proof.* For each  $X \in \mathcal{U}(\mathfrak{g})$  and  $\lambda \in F^*$  (the topological dual of  $F$ ), define the function  $f_{X,\lambda}: \widetilde{G}_r \rightarrow \mathbb{C}$ :

$$f_{X,\lambda}(g) = \lambda(\pi_0(g)\rho(X)v)$$

Let  $\lambda$  be such that  $\lambda = 0$  on  $\overline{\mathcal{U}(\mathfrak{g})v}$ . It is easily checked that the infinitesimal action of  $\mathcal{U}(\mathfrak{g}_{0,r})$  preserves the analytic vectors hence we obtain

$$Zf_{X,\lambda}(1_G) = 0 \quad \text{for each } Z \in \mathcal{U}(\mathfrak{g}_0)$$

Since  $\widetilde{G}_r$  is connected, we conclude that  $f_{X,\lambda} = 0$  on  $\widetilde{G}_r$ . Hence by the Hahn-Banach theorem we conclude that  $\text{span}\{\pi_0(G)\rho(X)v\}$  is contained in  $\overline{\mathcal{U}(\mathfrak{g})v}$ . Since this is true for all  $X \in \mathcal{U}(\mathfrak{g})$  we conclude that  $\overline{\mathcal{U}(\mathfrak{g})v}$  is  $\widetilde{G}_r$  invariant. Since

$$\rho(Y)\pi_0(g)\rho(X)v = \pi_0(g)\rho((g^{-1}Y)X)v$$

it is also clear that it is the smallest  $G_r$ -invariant subspace of  $F$  containing  $v$ . ■

**Theorem 3.36.** *Let  $S = G_r B^+ / B^+$  and assume  $F^1 := L^\chi(S) \neq 0$  modulo  $J$  the submodule generated by the odd part. Then:*

1.  $F^1$  contains an element  $\psi$  which is an analytic continuation of  $1^\sim$ ;
2.  $F^{11} := \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi} \subset L^\chi(S)$  is a Fréchet  $G_r$ -module,  $K_r$ -finite and with  $K_r$ -finite part  $\ell(\mathcal{U}(\mathfrak{g}))\psi = \mathcal{P}_\lambda^\sim$ .
3. the  $K_r$ -finite part  $\mathcal{P}_\lambda^\sim$  is isomorphic to  $\pi_{-\lambda}$  the irreducible representation with lowest weight  $-\lambda$ . In particular  $\lambda(H_\alpha) \in \mathbf{Z}_{\geq 0}$  for all compact positive roots  $\alpha$ .

*Proof.* We first establish the  $K_r$ -finiteness of  $F^1$ . From (2) of Cor. 3.18 it follows that the subspace  $F^1(\tau)$  injects (through the restriction morphism) in  $F^2(\tau)$ , where  $F^2$  denotes  $L^\chi(\Gamma_2)$  (we recall that  $\Gamma_2 = (G_r B^+ \cap \Gamma)^0 / B^+$ ). From (3) of Cor. 3.18, we know that  $\dim F^2(\tau) = \dim F(\tau)$ . By (3) of Corollary 3.17 and (2) of Corollary 3.15, we finally obtain  $\dim F(\tau) < +\infty$ . Hence  $F^1$  is  $A_r$ -finite. By Corollary 2.41 the  $A_r$  finiteness implies in our case the  $K_r$ -finiteness. Hence  $F^1$  is  $K_r$ -finite

We now go to the proof of (1). Assume that the  $K_r$ -finite part  $(F^1)^0 = \sum F^1(\tau)$  does not include the weight  $-\lambda$ , in other words we assume there is no analytic continuation of  $1^\sim$  to  $S$ . By Corollary 3.17 and Corollary 3.18,  $(F^1)^0$  is isomorphic to a subset of the set of polynomials in the  $t_\alpha$  (see Sec. 3.4 for the notation). Since  $1^\sim \in F^2(-\lambda) \supseteq F^1(-\lambda)$  we have that all the elements  $f$  in  $(F^1)^0$  are zero when evaluated at  $1_G$ . Hence, by the density of  $(F^1)^0$  in  $F^1$ , all the elements in  $F^1$  vanish at  $1_G$ . Using the  $G_r$  action it follows that  $\tilde{f} = 0$  for all  $f \in F^1$ . Hence  $F^1 = 0$  modulo  $J$  contradicting our hypothesis.

As for (2),  $F^{11} := \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi}$  is a Frechét superspace, since it is a closed subspace of a Frechét superspace. The fact that  $F^{11}$  is a  $G_r$ -module follows from Lemma 3.35. Hence  $F^{11}$  is a  $G_r$  submodule of  $F^1$ , and it is  $K_r$ -finite since it is a submodule of the  $K_r$  finite module  $F^1$ .  $\mathcal{J} = \ell(\mathcal{U}(\mathfrak{g}))\psi$  is clearly a highest weight  $\mathcal{U}(\mathfrak{g})$  module. Since  $F^{11}$  is the closure of the  $K_r$ -finite subspace  $\mathcal{J}$  subspace, its  $K_r$ -finite part is precisely  $\mathcal{J}$ .

(3). We know that  $\mathcal{J} \subset (F^1)^0 \hookrightarrow (F^2)^0 \simeq F^0$ . Clearly  $\mathcal{J} \hookrightarrow \mathcal{I}^\sim := \mathcal{U}(\mathfrak{g})1^\sim \subset F^0$ , but since by 3.26  $\mathcal{I}^\sim$  is irreducible, we have  $\mathcal{J} = \mathcal{I}^\sim$ .  $\mathcal{J}$  is the irreducible lowest weight module of lowest weight  $-\lambda$  or equivalently  $\mathcal{J}$  is the irreducible highest weight module of highest weight  $-\lambda$  with respect to the positive system  $-P$ . The  $K_r$ -finiteness of  $\mathcal{J}$  implies that  $-\lambda(H_{-\alpha}) \geq 0$ , hence our result. ■

**Corollary 3.37.** *Let the notation be as above. Then  $\mathcal{J} = \mathcal{U}(\mathfrak{g})\psi \subset (F^1)^0$  is the irreducible Harish-Chandra module with highest weight  $-\lambda$  with respect to the positive system  $-P$ .*

*Proof.* This is an immediate consequence of Proposition 3.33. ■

The shortcoming of the previous theorem is that it does not tell us when  $L^\chi(S) \neq 0$ . We shall take care of this after a remark relating our theory

with the Borel-Weil-Bott result and some observations regarding admissible systems.

**Remark 3.38.** Assume we are in the ordinary setting. Suppose that the real form  $G_r$  is actually the maximal compact subgroup  $U_r$ , i.e.,  $G_r = U_r$ . Then  $S = UB = G$ , so that  $L^\chi(S)$  is the space of global sections of the holomorphic bundle. In this case if we are given that we have a holomorphic finite dimensional representation with highest weight  $\lambda$  where  $\lambda$  is integral and dominant, i.e.,  $\lambda(H_\alpha) \geq 0$  for all positive roots (here all roots are compact as  $G_r = U_r$ ), then we can proceed (in the classical setting) using the Borel-Weil-Bott theorem and the concept of matrix element.

Before we state one of our main results we need few observations, which do not appear in this form in [10], but they are immediate consequences of the results in there and the ordinary theory. These considerations guarantee that when we choose an admissible system, we can construct infinite dimensional  $(\mathfrak{g}, \mathfrak{k})$ -representations

**Observation 3.39.** Let  $V_0$  be a highest weight  $(\mathfrak{g}_0, \mathfrak{k}_0)$ -module (purely even) with respect to the positive system  $Q$  (not necessarily admissible) of highest weight  $\lambda$ . Then  $\lambda(H_\gamma) \in \mathbf{Z}_{\geq 0}$  for all positive even roots  $\gamma$  which are not totally positive. In particular if  $Q$  is not admissible and  $V_0$  is irreducible, then  $V_0$  is finite dimensional. Hence if we have an admissible system for  $\mathfrak{g}$ , we can have  $\lambda(H_\gamma) \notin \mathbf{Z}_{\geq 0}$  for some non compact  $\gamma$ , hence the corresponding highest weight  $(\mathfrak{g}, \mathfrak{k})$ -module (super) is infinite dimensional. The existence of an admissible systems for  $\mathfrak{g}$ , that is the existence of totally positive roots, allows infinite dimensional representations.

**Definition 3.40.** We say that a dominant integral weight  $\lambda$  is of  $K$ -type if the  $\mathfrak{k}$  irreducible representation associated with  $\lambda$  can be lifted to  $K$ .

Now comes one of the main results of this paper.

**Theorem 3.41.** *Let the notation be as above. Assume the following:*

- $\dim(\mathfrak{c}) \geq 1$ .
- $\lambda \in \mathfrak{h}^*$  is integral and  $\lambda(H_\alpha) \geq 0$  for  $\alpha$  compact positive root.
- $\lambda$  is of  $K$ -type.

Then

1.  $L^\chi(G_r B^+ / B^+) \neq 0$ ;
2.  $F_\lambda^{11}$  is a  $G_r$  representation whose  $K_r$ -finite part is the lowest weight representation  $\pi_{-\lambda}$ .

*Proof.* Enough to show that  $L^\chi(G_r B^+) \neq 0$ , since (2) is an immediate consequence of Theorem 3.36. Let  $\sigma_\lambda$  be the finite dimensional irreducible representation of  $K_r$  with highest weight  $\lambda$  on the vector space  $V$ . Let  $v_\lambda$  be the corresponding highest weight vector. We can define the coefficient of the representation  $\sigma$  corresponding to  $v_\lambda$  that is the nonzero section  $a_{11} : K_r \rightarrow \mathbb{C}$ ,  $a_{11}(k) = (\sigma_\lambda(k)v_\lambda)_{v_\lambda}$ , that is the  $v_\lambda$  component of  $\sigma_\lambda(k)v_\lambda$  corresponding to the weight decomposition of  $V$ . Using Proposition 3.28, we can extend  $a_{11}$  to a nonzero section in  $\mathcal{O}(P^- K P^+)$ . Since  $G_r$  is embedded into  $P^- K P^+$  and  $a_{11}(1_G) = 1$  we obtain a non zero section of  $G_r$ , that is  $a_{11} \in \mathcal{O}(G_r)$ . It is immediate to verify:

$$r_b^* a_{11} = \chi_0^\lambda(b)^{-1} a_{11}, \quad b \in \widetilde{B}^+, \quad D_X^L a_{11} = -\lambda(X) a_{11}, \quad X \in \mathfrak{b}^+$$

so that  $a_{11} \in L^\chi(S)$  as requested. ■

**Remark 3.42.** The condition  $\dim(\mathfrak{c}) \geq 1$  is a necessary condition to have an infinite dimensional highest weight representation (see 3.27 and 3.39).

## 4 The Siegel Superspace

In this section we want to provide an explicit example of the theory of supersymmetric spaces we have developed so far. We start with a concise account of the ordinary setting.

### 4.1 Lagrangian manifold and Siegel space

We briefly summarize the results for the ordinary Siegel space. For more details see [26] Ch. VIII, Sec. 4.

The symplectic group consists of the matrices preserving a skewsymmetric form:

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) \mid A^t J_0 A = J_0\}, \quad J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where  $I_n$  is the identity matrix of rank  $n$ . We have  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C})$  if and only if  $a^t c$ ,  $b^t d$  symmetric and  $a^t d - c^t b = I_n$ , where  $a, b, c, d$  are  $n \times n$  matrices. The Lie algebra  $\mathfrak{sp}_{2n}(\mathbb{C}) = \mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{C}))$  is:

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{X \in \mathrm{GL}_{2n}(\mathbb{C}) \mid X^t J_0 + J_0 X = 0\} = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \mid b, c \text{ symmetric} \right\}$$

and  $\dim \mathfrak{sp}_{2n}(\mathbb{C}) = \dim \mathrm{Sp}_{2n}(\mathbb{C}) = 2n^2 + n$ .  $\mathrm{Sp}_{2n}(\mathbb{C})$  acts transitively on the set of *Langrangian spaces*.

**Definition 4.1.** A subspace  $L \subset \mathbb{C}^{2n}$  of dimension  $n$  is called *lagrangian* if

$$u^t J_0 v = 0, \quad \text{for all } u, v \in L.$$

The *lagrangian*  $\mathcal{L}_0$  is the set of lagrangian subspaces.

$\mathcal{L}_0$  is a subset of the grassmannian  $\mathrm{Gr}(n, 2n)$  of  $n$  spaces in  $\mathbb{C}^{2n}$ . We can represent a subspace  $W$  of dimension  $n$  with a  $2n \times n$  matrix whose columns form a basis for  $W$ :

$$W \cong \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ with } Z_i \in \mathrm{M}_n(\mathbb{C}), \text{ such that } \mathrm{rank} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = n. \quad (27)$$

This representation is not unique: two matrices  $(Z_1, Z_2)$  and  $(Y_1, Y_2)$  define the same lagrangian space if and only if there is a  $C \in \mathrm{GL}_n(\mathbb{C})$  such that  $Z_i C = Y_i$  ( $i = 1, 2$ ); hence we have a natural right action of  $\mathrm{GL}_n(\mathbb{C})$  on the set of  $2n \times n$  matrices of rank  $n$ ,  $\mathrm{M}_{2n \times n}(\mathbb{C})$ .

In terms of the representation (27), the condition that  $W$  is lagrangian translates into the equation

$$Z_1^t Z_2 - Z_2^t Z_1 = 0 \quad (28)$$

that is,  $W$  is lagrangian if and only if  $Z_1^t Z_2$  is symmetric. Hence  $\mathcal{L}_0$  is the subset of  $\mathrm{Gr}(n, 2n)$ , the grassmannian of  $n$  subspaces in  $\mathbb{C}^{2n}$ , defined by the

quadratic relations (28), where we identify  $Gr(n, 2n)$  with  $M_{2n \times n}(\mathbb{C})/GL_n(\mathbb{C})$ . Consequently  $\mathcal{L}_0$  is a closed subvariety of  $Gr(n, 2n)$ , hence it is a complex projective variety.

In  $\mathcal{L}_0$  we consider:

$$\begin{aligned}\mathcal{L}_{0,f} &= \left\{ \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \mid Z_1^t Z_2 \text{ symmetric, } \det(Z_2) \neq 0 \right\} / GL_n(\mathbb{C}) \\ &\cong \left\{ \begin{pmatrix} Z \\ 1 \end{pmatrix} \mid Z \text{ symmetric} \right\} = \{X + iY \mid X, Y \in M_{2n \times n}(\mathbb{R}), \text{ symmetric}\}.\end{aligned}$$

There is a natural transitive left action of  $Sp_{2n}(\mathbb{C})$  on  $\mathcal{L}_0$ :

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \mapsto \begin{pmatrix} aZ_1 + bZ_2 \\ cZ_1 + dZ_2 \end{pmatrix}.$$

where  $a, b, c, d$  are  $n \times n$  matrices with complex entries. This action gives the identification

$$\mathcal{L}_0 \cong Sp_{2n}(\mathbb{C})/P_0, \quad P_0 = \left\{ \begin{pmatrix} a & b \\ 0 & (a^t)^{-1} \end{pmatrix} \mid a^{-1}b \text{ symmetric} \right\} \subset Sp_{2n}(\mathbb{C})$$

$P_0$  is the closed subgroup of  $Sp_{2n}(\mathbb{C})$  which stabilizes  $\langle e_1, \dots, e_n \rangle$ , where  $\{e_i\}_{1 \leq i \leq 2n}$  is the canonical basis of  $\mathbb{C}^{2n}$ . The action of  $Sp_{2n}(\mathbb{C})$  on  $\mathcal{L}_{0,f}$  (which is not stable) reads as:

$$Z \mapsto (aZ + b)(cZ + d)^{-1}, \quad Z \text{ symmetric}$$

These are the so called *generalized linear fractional transformations*.

**Definition 4.2.** We define the *Siegel space* as:

$$\mathcal{S}_0 = \{Z \mid Z = X + iY, X, Y \in M_{2n \times n}(\mathbb{R}) \text{ symmetric, } Y > 0\} \subset \mathcal{L}_{0,f}.$$

**Proposition 4.3.**  $Sp_{2n}(\mathbb{R})$  acts transitively on  $\mathcal{S}_0$  and the stabilizer of  $iI$  is  $U(n)$ , the maximal compact in  $Sp_{2n}(\mathbb{R})$ . Hence:

$$\mathcal{S}_0 \cong Sp_{2n}(\mathbb{R})/U(n).$$

*Proof.* See [26] Ch. VIII Sec. 7. ■

Notice that by definition,  $\mathcal{S}_0$  is open in  $\mathcal{L}_{0,f}$ , with respect to the complex topology, so it is naturally a complex manifold. This fact is interesting since one may define alternatively  $\mathcal{S}_0$  using Proposition 4.3 as  $\mathcal{S}_0 \cong \mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}(n)$ . With this definition  $\mathcal{S}_0$  is a real manifold, and it is not evident it has a natural invariant complex structure. It was Harish-Chandra [25] to understand the realization of the symmetric domains, like  $\mathcal{S}_0$ , in such a way that this structure became manifested naturally.

We shall pursue in the next section this important point of view directly in the supersetting.

## 4.2 The orthosymplectic supergroup

The orthosymplectic supergroup  $\mathrm{Osp}(V)$  is the supergroup preserving a certain supersymmetric bilinear form on the vector superspace  $V$ . If we take  $V = k^{m|n}$  ( $k = \mathbb{R}$  or  $\mathbb{C}$ ) and the standard supersymmetric bilinear form given in the canonical basis by

$$J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

we can formulate the following definition (see also [8] Ch. 7, 8, 11 for more details).

**Definition 4.4.** We define *orthosymplectic supergroup* the group valued functor:

$$(\mathrm{smflds})_k^o \longrightarrow (\mathrm{groups})$$

$$T \longmapsto \mathrm{Osp}(m|2n, k)(T)$$

$$\mathrm{Osp}(m|2n, k)(T) := \{A \in \mathrm{GL}(m|2n, k)(T) \mid A^t J A = J\},$$

where  $(\mathrm{smflds})_{\mathbb{C}}$  (resp.  $(\mathrm{smflds})_{\mathbb{R}}$ ) is the category of complex analytic (resp. differentiable) supermanifolds and  $(\mathrm{groups})$  the category of groups.  $\mathrm{GL}(m|2n, k)$  is the general linear supergroup over the field  $k$ ; we may omit  $k$  when the meaning is clear.  $I_m$  is the identity matrix of rank  $m$  and  $A^t$  denotes the super transpose (see [43] pg 293). When we take  $k = \mathbb{C}$  we speak of the complex orthosymplectic supergroup, denoted simply as  $\mathrm{Osp}(m|2n)$ , whenever there is no danger of confusion. When we take  $k = \mathbb{R}$  we speak of the real orthosymplectic supergroup and we denote it with  $\mathrm{Osp}(m|2n, \mathbb{R})$ .

These functors are representable and they correspond to analytic and differentiable supergroups respectively (see [43] pg 293).

As usual, since  $\text{Osp}(m|n, k)$  is a subfunctor of  $\text{GL}(m|2n, k)$ , we omit its definition on the arrows, since it is the same as for  $\text{GL}(m|2n, k)$ . We will do the same also for all of the functors for which the definition of the arrows is coming from the one for  $\text{GL}(m|2n, k)$ .

Let us take now  $k = \mathbb{C}$ . Let  $A = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \text{GL}(m|2n)(T)$ . We have  $A \in \text{Osp}(m|2n)(T)$  if and only if:

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^t J \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \begin{pmatrix} a^t & \beta^t \\ -\alpha^t & b^t \end{pmatrix} J \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = J$$

which gives:

$$\begin{aligned} a^t a + \beta^t J_0 \beta &= 1 & a^t \alpha + \beta^t J_0 b &= 0 \\ -\alpha^t a + b^t J_0 \beta &= 0 & -\alpha^t \alpha + b^t J_0 b &= J_0 \end{aligned} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

More explicitly taking:

$$A = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} \quad (29)$$

where  $b_{ij}$  are  $n \times n$  matrices,  $a$  is a  $m \times m$  matrix, we have the relations:

$$\begin{aligned} a^t a + \beta_2^t \beta_1 - \beta_1^t \beta_2 &= 1 & \alpha_2^t \alpha_2 + b_{12}^t b_{22} - b_{22}^t b_{12} &= 0 \\ a^t \alpha_1 + \beta_2^t b_{11} - \beta_1^t b_{21} &= 0 & a^t \alpha_2 + \beta_2^t b_{12} - \beta_1^t b_{22} &= 0 \\ \alpha_1^t \alpha_1 + b_{11}^t b_{21} - b_{21}^t b_{11} &= 0 & \alpha_1^t \alpha_2 + b_{11}^t b_{22} - b_{21}^t b_{12} &= 1 \end{aligned} \quad (30)$$

We now compute the Lie superalgebra, using the method described in [8].

$$\text{Lie}(\text{Osp}(m|2n)) = \{A \in \text{M}(m|2n) \mid (I + \epsilon A)^t J (I + \epsilon A) = J\}$$



Using for  $A$  the same notation as in (29)

$$\begin{pmatrix} 1 + \epsilon(a^t + a) & \epsilon(\alpha_1 + \beta_2^t) & \epsilon(\alpha_2 - \beta_1^t) \\ -\epsilon(\alpha_1^t + \beta_2) & \epsilon(b_{21} - b_{21}^t) & -1 + \epsilon(b_{11}^t + b_{22}) \\ -\epsilon(\alpha_1^t - \beta_1) & 1 - \epsilon(b_{22}^t + b_{11}) & \epsilon(b_{12}^t - b_{12}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence:

$$\text{Lie}(\text{Osp}(m|2n)) = \left\{ A = \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \alpha_2^t & b_{11} & b_{12} \\ -\alpha_1^t & b_{21} & -b_{11}^t \end{pmatrix} \mid a^t = -a, b_{21} = b_{21}^t, b_{12} = b_{12}^t \right\}$$

and

$$\dim \text{Lie}(\text{Osp}(m|2n)) = \frac{m(m-1)}{2} + 2n^2 + n|2mn$$

### 4.3 The Siegel superspace

In this section we introduce the Siegel superspace. For all of our notation and for the basic definitions of supergeometry, we refer the reader to [8], ch. 3, 4, 5.

Let  $k = \mathbb{C}$ . Consider the closed complex analytic subsupergroup  $P$  of the complex supergroup  $\text{Osp}(m|n)$  defined, via its functor of points as:

$$P(T) = \left\{ \begin{pmatrix} a & 0 & \alpha_2 \\ b_{11}\alpha_2^t a & b_{11} & b_{12} \\ 0 & 0 & (b_{11}^t)^{-1} \end{pmatrix} \mid \begin{cases} a^t a = 1 \\ (b_{11}^{-1} b_{12}) - (b_{11}^{-1} b_{12})^t = \alpha_2^t \alpha_2 \end{cases} \right\}$$

As for any closed analytic subsupergroup of an analytic Lie supergroup it is possible to construct the quotient  $\text{Osp}(m|2n)/P$ . This is an analytic supermanifold, its functor of points is the sheafification of the functor:

$$T \longmapsto \text{Osp}(m|2n)(T)/P(T), \quad T \in (\text{smflds})_{\mathbb{C}}$$

(for more details on the construction of quotients see [8] ch. 9).

**Definition 4.5.** We define the *super Lagrangian*  $\mathcal{L}$  as the complex supermanifold:

$$\mathcal{L} = \text{Osp}(m|2n)/P$$

Notice that, by the very definition, the reduced manifold of  $\mathcal{L}$  is  $\mathcal{L}_0$ , and we have a natural transitive action of  $\text{Osp}(m|2n)$  on the supermanifold  $\mathcal{L}$ . We also define  $\mathcal{L}_f$  as the open submanifold of  $\mathcal{L}$  corresponding to the open subset  $\mathcal{L}_{0,f}$  of  $\mathcal{L}_0$  (see Sec. 4.1).

**Remark 4.6.**  $\mathcal{L}$  is the complex supermanifold of Lagrangian subspaces in  $\mathbb{C}^{m|2n}$ . It is realized as the orbit of the topological point:

$$W_0 = \langle 0, \dots, 0, \epsilon_1, \dots, \epsilon_n, 0, \dots, 0 \rangle,$$

under the action of the complex orthosymplectic supergroup  $(e_i, \epsilon_j \mid i = 1, \dots, m, j = 1, \dots, 2n)$  is the canonical basis of  $\mathbb{C}^{m|2n}$ ). In fact, as one can readily check,  $P = \text{Stab} W_0$ .  $\mathcal{L}$  is a closed subsupermanifold of  $\text{Gr}(0|n; m|2n)$  and, as a matter of fact, also a subsupervariety of the algebraic supervariety  $\text{Gr}(0|n; m|2n)$ .

We now want to characterize the functor of points of  $\mathcal{L}_f$ . We start with a lemma.

**Lemma 4.7.** *We can always choose uniquely a representative of the class*

$$\begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T) \in \mathcal{L}_f$$

*in the form*

$$\begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad (31)$$

*Proof.* We need to show that for each  $g \in \text{Osp}(m|n)$ , there exists an element  $p \in P(T)$  such that  $gp = (z, \zeta)$ , where  $(z, \zeta)$  is a shorthand to express the element in the form (31). It is actually easier, but equivalent, to show that  $(z, \zeta)^{-1}g \in P(T)$ , which amounts to find  $z, \zeta, u, v, w, \xi$  depending on  $\alpha_i, \beta_i$  and  $a, b_{ij}$  in the following equation:

$$\begin{pmatrix} 1 & 0 & -\zeta \\ 0 & 0 & 1 \\ \zeta^t & -1 & z - \zeta^t \zeta \end{pmatrix} \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} u & 0 & \xi \\ v \xi^t u & v & w \\ 0 & 0 & (v^t)^{-1} \end{pmatrix}$$

Notice that there is no loss of generality in assuming  $b_{21} = 1$ . The check the solutions are unique and compatible with the conditions defining  $\text{Osp}(m|2n)$  is a direct calculation. The values obtained are:

$$u = a - \alpha_1 \beta_2, \quad \xi = \alpha_2 - \alpha_1 b_{22}, \quad v = 1, \quad w = b_{22}, \quad z = b_{11}, \quad \zeta = \alpha_1$$

■

**Proposition 4.8.** *The  $T$ -points of the supermanifold  $\mathcal{L}_f$  are identified with the matrices in  $\text{Osp}(m|n)(T)$  of the form:*

$$\mathcal{L}_f(T) \simeq \left\{ \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \mid \zeta^t \zeta + z^t - z = 0 \right\}$$

Hence

$$\mathcal{L}_f \cong \mathbb{C}^{\frac{n^2+n}{2}|mn}.$$

*Proof.* Let us choose a suitable open cover  $\{T_i\}_{i \in I}$  of  $T$ , so that

$$\mathcal{L}(T_i) = (\text{Osp}(m|2n)/P)(T_i) = \text{Osp}(m|2n)(T_i)/P(T_i).$$

We can then write

$$\mathcal{L}_f(T_i) = \left\{ \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T) \mid b_{21} \text{ invertible} \right\}$$

By the previous lemma, we can write:

$$\begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T_i) = \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} P(T_i), \quad \zeta^t \zeta + z^t - z = 0 \quad (32)$$

$\mathcal{L}_f$  is then defined by  $n(n-1)/2$  equations in  $\mathbb{C}^{n^2|mn}$ :

$$\sum_k \zeta_{ki} \zeta_{kj} + z_{ji} - z_{ij} = 0, \quad 1 \leq i < j \leq n$$

Hence the result follows. ■

**Remark 4.9.** The fact  $\mathcal{L}_f$  is a supermanifold defined by non linear equations marks an important difference with the classical setting (refer to Sec. 4.1).

**Definition 4.10.** We define *Siegel superspace* the open supermanifold of  $\mathcal{L}_f$  corresponding to the complex open subset  $\mathcal{S}_0$  (see Sec. 4.1).

By the Chart Theorem (see ch. 4 in [8]) we have that a  $T$ -point of the Siegel superspace  $\mathcal{S}$  corresponds to a choice of two matrices  $\zeta$  and  $z$  with

entries in  $\mathcal{O}(T)$  such that their values at all topological points of  $|T|$  land in  $S_0$ . In other words,  $\mathcal{S}(T)$  consists of the following elements in  $\mathcal{L}_f(T)$ :

$$\mathcal{S}(T) = \left\{ \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \mid \begin{cases} \zeta^t \zeta + z^t - z = 0 \\ z = x + iy, \tilde{y}(t) > 0, \forall t \in |T| \end{cases} \right\} \subset \mathcal{L}_f(T)$$

Notice that the condition  $\tilde{y}(t) > 0$  is meaningful since the defining equation  $\zeta^t \zeta + z^t - z = 0$ , when evaluated at the topological points implies that  $\tilde{y}$ , is a symmetric real matrix.

From now on we denote the  $T$ -points  $\mathcal{L}_f(T)$  and  $\mathcal{S}(T)$  either using matrices (see Prop. 4.8) or as  $(z, \zeta) \in \mathbb{C}^{n^2|mn}(T)$ , where  $z$  and  $\zeta$  are restricted by the conditions detailed above.

We now want to realize the Siegel superspace as a real homogeneous supermanifold. Observe first that any complex supergroup  $G \subset \mathrm{GL}(m|n, \mathbb{C})$  can be viewed naturally as a real supergroup and a real form of  $G$  corresponds to an involution of its  $T$ -points (see [22] Ch. 1 for more details). Within this framework, we interpret  $\mathrm{Osp}(m|2n, \mathbb{R})$  as the real supergroup corresponding to the involution:

$$\mathrm{Osp}(m|2n, \mathbb{C})(T) \xrightarrow{\rho_T} \mathrm{Osp}(m|2n, \mathbb{C})(T), \quad \begin{pmatrix} t & \theta \\ \eta & s \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{t} & \bar{\theta} \\ \bar{\eta} & \bar{s} \end{pmatrix}.$$

where  $t = (t_{ij})$ ,  $t_{ij} = t_{ij}^1 + it_{ij}^2$ ,  $\bar{t} = (\bar{t}_{ij})$ ,  $\bar{t}_{ij} = t_{ij}^1 - it_{ij}^2$  and similarly for  $\bar{\theta}$ ,  $\bar{\eta}$  and  $\bar{s}$ .  $T$  is a real supermanifold and

$$\mathrm{Osp}(m|2n, \mathbb{C})_{\mathbb{R}}(T) = \mathrm{Hom}(\mathcal{O}(\mathrm{Osp}(m|2n, \mathbb{C})), \mathcal{O}(T) \otimes \mathbb{C})$$

Here  $\mathrm{Osp}(m|2n, \mathbb{C})_{\mathbb{R}}$  denotes the complex supergroup  $\mathrm{Osp}(m|2n, \mathbb{C})$  viewed as a real supergroup (see also Sec. 2.2). So we can identify an element in  $\mathrm{Osp}(m|2n, \mathbb{C})_{\mathbb{R}}(T)$  as a matrix in the form specified above. Hence

$$\mathrm{Osp}(m|2n, \mathbb{R})(T) \subset \mathrm{Osp}(m|2n, \mathbb{C})_{\mathbb{R}}(T)$$

.

Consider the natural action of  $\mathrm{Osp}(m|2n, \mathbb{R})$  on the quotient  $\mathrm{Osp}(m|2n)/P$  and restrict it to  $\mathcal{S}$ :

$$\mathrm{Osp}(m|2n, \mathbb{R})(T) \times \mathcal{S}(T) \longrightarrow \mathcal{S}(T)$$

$$g, \begin{pmatrix} 1 & \zeta & 0 \\ -\zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \frac{a\zeta + \alpha_1 z + \alpha_2}{\beta_2 \zeta + b_{21} z + b_{22}} & 0 \\ \left( \frac{a\zeta + \alpha_1 z + \alpha_2}{\beta_2 \zeta + b_{21} z + b_{22}} \right)^t & \frac{\beta_1 \zeta + b_{11} z + b_{12}}{\beta_2 \zeta + b_{21} z + b_{22}} & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\text{For } g = \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} \in \mathrm{Osp}(m|2n, \mathbb{R}).$$

This is well defined, because the ordinary action is well defined on the classical Siegel space, hence, since the denominators appearing in the super-setting differ by nilpotent elements from the classical ones, they will still be invertible on the  $T$ -points of the Siegel superspace.

In the  $(z, \zeta)$  coordinates for the supermanifold  $\mathcal{S}$ , the action of  $\mathrm{Osp}(m|2n, \mathbb{R})(T)$  on  $\mathcal{S}$  is (more appropriately):

$$g \cdot (z, \zeta) = ((\beta_1 \zeta + b_{11} z + b_{12})(\beta_2 \zeta + b_{21} z + b_{22})^{-1}, (a\zeta + \alpha_1 z + \alpha_2)(\beta_2 \zeta + b_{21} z + b_{22})^{-1})$$

**Theorem 4.11.**  *$\mathrm{Osp}(m|2n, \mathbb{R})$  acts transitively on the Siegel superspace and the stabilizer of the topological point  $(iI, 0) \in |\mathcal{S}|$  is the subgroup:*

$$K_{\mathbf{r}}(T) = \mathrm{Stab}(iI, 0) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & -b_{12} & b_{11} \end{pmatrix} \in \mathrm{Osp}(m|2n, \mathbb{R})(T) \right\}$$

which is compact and coincides with its reduced group:  $(K_{\mathbf{r}})_{\mathrm{red}} = K_{\mathbf{r}}$  and it is equal to  $\mathrm{O}(m) \times \mathrm{U}(n)$ . So we have the isomorphism as real supermanifolds:

$$\mathcal{S} \cong \mathrm{Osp}(m|2n, \mathbb{R})/K_{\mathbf{r}}.$$

*Proof.* The action of  $|\mathrm{Osp}(m|2n, \mathbb{R})|$  on  $|\mathcal{S}|$  is transitive (see Sec. 4.1). Consider the supermanifold morphism  $a_p : \mathrm{Osp}(m|2n, \mathbb{R}) \longrightarrow \mathcal{S}$ ,  $a_p(g) = g \cdot (0, iI)$ . The differential  $(da_p)_I$  at the identity is surjective, hence the result follows by Obs. 2.12 and Prop. 2.14 (see also Prop. 9.1.4 in [8]). ■

## 4.4 The super Cayley transform

The form we have found for  $K_{\mathbf{r}}$  is not suitable for Lie superalgebra calculations, so we need to transform  $\mathcal{S}$ , so that also  $K_{\mathbf{r}}$  transforms accordingly. We shall do this via the *super Cayley transform*. Our aim is to find a real form for  $\text{Osp}$  which enables us to relate our construction to the results in [10] about admissible systems.

Classically the Cayley transform maps the unit disc  $\mathcal{D}_0^1 \subset \mathbb{C}$  isomorphically onto the classical Siegel upper half plane  $\mathcal{S}_0^1$ :

$$\begin{array}{ccc} \mathcal{D}_0^1 & \longrightarrow & \mathcal{S}_0^1 \\ z & \mapsto & i(1+z)/(1-z) \end{array}$$

This morphism can be generalized to more than one dimension (see [30] VI, Sec. 7). We now want to take a step forward and generalize it to the Siegel superspace.

Consider the following linear transformation:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/\sqrt{2i} & i/\sqrt{2i} \\ 0 & -1/\sqrt{2i} & 1/\sqrt{2i} \end{pmatrix} \in \text{Osp}(m|2n)(\mathbb{C}) \subset \text{Osp}(m|2n)(T)$$

with  $\sqrt{2i} = e^{i\pi/4}$ . Furthermore: to ease an otherwise heavy notation, we shall not write the identity matrix or write 1 in place of it.

Define the open subsupermanifold  $\mathcal{D} = (\mathcal{D}_0, \mathcal{O}_{\mathcal{L}_f|_{\mathcal{D}_0}})$  of  $\mathcal{L}_f$  with topological space:

$$\mathcal{D}_0 = \left\{ \begin{pmatrix} 0 \\ z \\ 1 \end{pmatrix} \mid z \in M_{n \times n}(\mathbb{C}) \text{ symmetric, } 1 - z\bar{z} > 0 \right\}$$

**Proposition 4.12.** *The linear transformation  $L$  induces a supermanifold diffeomorphism:*

$$\begin{array}{ccc} \phi_T : \mathcal{D}(T) & \longrightarrow & \mathcal{S}(T) \\ \begin{pmatrix} \eta \\ z \\ 1 \end{pmatrix} & \longmapsto & \begin{pmatrix} \sqrt{2i}\eta(1-z)^{-1} \\ i(z+1)(1-z)^{-1} \\ 1 \end{pmatrix} \end{array}$$

*Proof.* Let us take a generic element in  $\mathcal{L}_f(T)$  and multiply it by  $L$ :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i/\sqrt{2i} & i/\sqrt{2i} \\ 0 & -1/\sqrt{2i} & 1/\sqrt{2i} \end{pmatrix} \begin{pmatrix} \eta \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ \frac{i}{\sqrt{2i}}(z+1) \\ \frac{1}{\sqrt{2i}}(1-z) \end{pmatrix} \sim \begin{pmatrix} \sqrt{2i}\eta(1-z)^{-1} \\ i(z+1)(1-z)^{-1} \\ 1 \end{pmatrix}$$

The map  $\phi$  can be extended on the lagrangian  $\mathcal{L}_f$  (except at the locus  $z = 1$ ) and it is differentiable on  $|\mathcal{L}_f| \setminus \{z = 1\}$ . Since  $|\phi|$  is an homeomorphism when restricted to  $|\mathcal{D}|$  and the differential  $d\phi$  is surjective, the result follows (see [8]).  $\blacksquare$

We call the diffeomorphism  $\phi$  the *super Cayley transform*.

We now want to define another real form of the orthosymplectic supergroup and for the compact  $K_{\mathbf{r}}$ , so to realize also  $\mathcal{D}$  as real homogeneous space. This point of view will be useful in the next section when we realize the Siegel superspace as a hermitian superspace. We start by defining the real Lie supergroup functors:

$$\mathrm{Osp}_{\mathcal{D}}(m|2n)(T) := L^{-1}\mathrm{Osp}(m|2n, \mathbb{R})(T)L, \quad K_{\mathcal{D}}(T) := L^{-1}K_{\mathbf{r}}(T)L$$

**Proposition 4.13.**  *$\mathrm{Osp}_{\mathcal{D}}(m|2n)$  is a real form of the orthosymplectic supergroup and its functor of points is explicitly given by:*

$$\mathrm{Osp}_{\mathcal{D}}(m|2n)(T) = \left\{ \begin{pmatrix} a_0 & \alpha_1 & -i\bar{\alpha}_1 \\ \beta_1 & b_{11} & b_{12} \\ i\bar{\beta}_1 & \bar{b}_{12} & \bar{b}_{11} \end{pmatrix} \right\} \subset \mathrm{Osp}(m|2n)(T)$$

where  $a_0 \in O(m)$ ,  $T$  is a real supermanifold and  $\mathrm{Osp}(m|2n)(T)$  is the complex orthosymplectic supergroup viewed as a real supergroup.

$K_{\mathcal{D}}$  is a real form of the compact group  $K_{\mathbf{r}}$  and it is given by

$$K_{\mathcal{D}} = \left\{ \begin{pmatrix} a_0 & 0 & 0 \\ 0 & b_{11} & 0 \\ 0 & 0 & \bar{b}_{11} \end{pmatrix} \mid b_{11}\bar{b}_{11}^t = 1, a_0 \in O(m) \right\}$$

*Proof.* We need to find a conjugation defining  $\mathrm{Osp}_{\mathcal{D}}(m|2n)$  inside  $\mathrm{Osp}(m|2n)$ . We notice that  $h \in \mathrm{Osp}_{\mathcal{D}}(m|2n)(T)$  if and only if  $LhL^{-1} \in \mathrm{Osp}(m|2n, \mathbb{R})(T)$  so we have:

$$(LhL^{-1})^t J (LhL^{-1}) = J, \quad (LhL^{-1})^t J (\bar{L}h\bar{L}^{-1}) = J$$

the second equation true since  $LhL^{-1}$  in the real form of  $\text{Osp}(m|2n)$  defined by complex conjugation. We obtain:

$$h^t J h = J, \quad h^t N \bar{h} = N, \text{ with } N := L^t J \bar{L}$$

Let  $F = J^{-1}N$ . By substituting in  $h^t N \bar{h} = N$  we get that  $h \in \text{Osp}_{\mathcal{D}}(m|2n)(T)$  if and only if  $h \in \text{Osp}(m|2n)(T)$  and  $h = F \bar{h} F^{-1}$ , where

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

The conjugation in  $\text{Osp}(m|2n)$  whose fixed points give  $\text{Osp}_{\mathcal{D}}(m|2n)$  is as follows:

$$\begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & -i\bar{\alpha}_2 & -i\bar{\alpha}_1 \\ i\bar{\beta}_2 & \bar{b}_{22} & \bar{b}_{21} \\ i\bar{\beta}_1 & \bar{b}_{12} & \bar{b}_{11} \end{pmatrix}$$

which gives immediately the claim. The statement about  $K_{\mathcal{D}}$  is entirely classical and known.  $\blacksquare$

**Proposition 4.14.**  *$\text{Osp}_{\mathcal{D}}(m|2n)$  acts transitively on  $\mathcal{D}$  and  $K_{\mathcal{D}}$  is the stabilizer of the topological point  $(1, 0)$ . Hence*

$$\mathcal{D} \cong \text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}.$$

*Proof.* It the same as for 4.11.  $\blacksquare$

We now compute the real Lie superalgebras of  $\text{Osp}_{\mathcal{D}}(m|2n)$  and  $K_{\mathcal{D}}$ .

**Proposition 4.15.** *We have that*

$$\begin{aligned} \text{osp}_{\mathcal{D}}(m|2n) &= \text{Lie}(\text{Osp}_{\mathcal{D}}(m|2n)) = \\ &= \left\{ \begin{pmatrix} x & \xi & -i\bar{\xi} \\ -i\bar{\xi}^t & y_{11} & y_{12} \\ -\xi^t & \bar{y}_{12} & \bar{y}_{11} \end{pmatrix} \mid y_{12} \text{ symmetric}, x = -\bar{x}^t, y_{11} = -\bar{y}_{11}^t \right\} \\ \mathfrak{k}_{\mathcal{D}} &= \text{Lie}(K_{\mathcal{D}}) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y_{11} & 0 \\ 0 & 0 & \bar{y}_{11} \end{pmatrix} \right\} \end{aligned}$$



Moreover, we have the Cartan decomposition  $\mathfrak{osp}_{\mathcal{D}}(m|2n) = \mathfrak{k}_{\mathcal{D}} \oplus \mathfrak{p}_{\mathcal{D}}$  with

$$\mathfrak{p}_{\mathcal{D}} = \left\{ \begin{pmatrix} 0 & \xi & -i\bar{\xi} \\ -i\bar{\xi}^t & 0 & y_{12} \\ -\xi^t & \overline{y_{12}} & 0 \end{pmatrix} \mid y_{12} \text{ symmetric}, \right\}$$

corresponding to the Cartan automorphism:

$$\begin{aligned} \theta : \mathfrak{osp}_{\mathcal{D}}(m|2n) &\longrightarrow \mathfrak{osp}_{\mathcal{D}}(m|2n) \\ X &\longmapsto -\bar{X}^t, & X \in \mathfrak{osp}(m|2n)_0 \\ X &\longmapsto -i\bar{X}^t, & X \in \mathfrak{osp}(m|2n)_1 \end{aligned}$$

*Proof.* The conjugation defining  $\mathfrak{osp}_{\mathcal{D}}(m|2n)$  is obtained as follows:  
 $X \in \mathfrak{osp}_{\mathcal{D}}(m|2n)$  if and only if  $h \in \mathfrak{osp}(m|2n)$  and  $F\bar{X} = XF$  with  $F$  as above. An easy calculation shows the result.  $\blacksquare$

## 4.5 Super hermitian symmetric structure

We want to show that the superdomain  $\mathcal{D}$ , isomorphic to the Siegel superspace, is an *Hermitian symmetric superspace*.

**Lemma 4.16.** *For the complex Lie superalgebra  $\mathfrak{osp}(m|2n)$  we have the admissible system  $P = P_k \cup P_{n,0} \cup P_{n,1}$ , where:*

$$P_{n,0} = \{\epsilon_1 \pm \epsilon_j \mid 1 < j \leq m\} \cup \{\epsilon_1\} \cup \{\delta_i + \delta_j \mid 1 \leq i, j \leq n\}$$

$$P_{n,1} = \{\delta_i \pm \epsilon_j \mid 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{\delta_i \mid 1 \leq i \leq n\}$$

$$P_k = \{\epsilon_i \pm \epsilon_j \mid 1 < i < j \leq m\} \cup \{\epsilon_i \mid 1 < i \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\}$$

with

$$\Pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1\}$$

the simple system with one simple non-compact even root  $\epsilon_1 - \epsilon_2$  and one non-compact simple odd root:  $\delta_n - \epsilon_1$ .

*Proof.* The fact  $\Pi$  is a simple system for  $\mathfrak{osp}(m|2n)$  is detailed in [27]. In order to check  $P$  is admissible we need to check the following:

- If  $\alpha, \beta \in P_k$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta \in P_k$ .
- If  $\alpha, \beta \in P_n$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta \in P_k$ .
- If  $\alpha \in P_k \cup -P_k, \beta \in P_n$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta \in P_n$ .

These are all straightforward calculations. ■

This gives immediately the following proposition.

**Proposition 4.17.** *The complex Lie superalgebra  $\mathfrak{osp}(m|2n)$  is the direct sum of three Lie subsuperalgebras:*

$$\mathfrak{osp}(m|2n) = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

$$\mathfrak{k} = \sum_{\alpha \in P_k \cup -P_k} \mathfrak{g}_\alpha, \quad \mathfrak{p}^+ = \sum_{\alpha \in P_n} \mathfrak{g}_\alpha, \quad \mathfrak{p}^- = \sum_{\alpha \in -P_n} \mathfrak{g}_\alpha.$$

where  $\mathfrak{k} = \mathbb{C} \otimes \text{Lie}(K_{\mathcal{D}})$  and  $\mathfrak{p}^+ \oplus \mathfrak{p}^- = \mathbb{C} \otimes \mathfrak{p}_{\mathcal{D}}$

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & 0 & \xi \\ \xi^t & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & -\eta & 0 \\ 0 & 0 & 0 \\ \eta^t & v & 0 \end{pmatrix} \right\}.$$

We can now express explicitly the Harish-Chandra decomposition for  $\text{Osp}(m|2n)$ , proven in Prop. 3.28.

Let  $P^-$  and  $P^+$  be the complex subsupergroups of the complex orthosymplectic supergroup  $\text{Osp}(m|2n)$  defined via their functor of points as:

$$P^+ = \left\{ \begin{pmatrix} 1 & 0 & \xi \\ \xi^t & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad P^- = \left\{ \begin{pmatrix} 1 & -\eta & 0 \\ 0 & 1 & 0 \\ \eta^t & v & 1 \end{pmatrix} \right\}$$

Most immediately  $\mathfrak{p}^\pm = \text{Lie}(P^\pm)$ . Notice that while in the ordinary setting we have that the groups  $P_0^\pm$  are abelian, in the supersetting, this is no longer true. In the functor of points we can express, for example for  $P^-$ , the multiplication as follows:

$$\begin{pmatrix} 1 & -\eta & 0 \\ 0 & 1 & 0 \\ \eta^t & v & 1 \end{pmatrix} \begin{pmatrix} 1 & -\eta' & 0 \\ 0 & 1 & 0 \\ \eta'^t & v' & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\eta - \eta' & 0 \\ 0 & 1 & 0 \\ \eta^t + \eta'^t & v + v' - \eta^t \eta' & 1 \end{pmatrix}$$

By Prop. 3.28 we have that the supermanifold  $P^-KP^+$  is open in  $\text{Osp}(m|2n)$  and such fact is instrumental to produce a complex hermitian structure on the real quotient  $\mathcal{D} = \text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}$ .

We now want to realize explicitly in our example such structure. By its very construction  $\mathcal{D}$  is a complex supermanifold and it has a natural action of  $\text{Osp}_{\mathcal{D}}(m|2n)$ , so we need to check the conditions (1) and (2) of Definition 3.2. Notice that condition (2) amounts to the equation (20). By Prop. 4.15 we have the Cartan decomposition  $\text{osp}_{\mathcal{D}}(m|2n) = \mathfrak{k}_{\mathcal{D}} \oplus \mathfrak{p}_{\mathcal{D}}$ . The equation (20) reads then

$$J \circ \text{ad}(X)|_{\mathfrak{p}_{\mathcal{D}}} = \text{ad}(X)|_{\mathfrak{p}_{\mathcal{D}}} \circ J$$

with  $J$  the almost complex structure at the identity coset,  $J : \mathfrak{p}_{\mathcal{D}} \longrightarrow \mathfrak{p}_{\mathcal{D}}$ , where we identify  $\mathfrak{p}_{\mathcal{D}} = T_{K_{\mathcal{D}}}(\text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}})$ .

**Proposition 4.18.** *Let  $J = \text{ad}(c)|_{\text{osp}_{\mathcal{D}}(m|n)_0} + \text{ad}(2c)|_{\text{osp}_{\mathcal{D}}(m|n)_1}$ , where  $c$  is the element in the center of  $\mathfrak{k}_{\mathcal{D}}$ :*

$$c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix}$$

*Then  $J|_{\mathfrak{p}_{\mathcal{D}}}$  defines a complex structure on  $\mathfrak{p}_{\mathcal{D}}$ , which gives to an hermitian structure on  $\text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}$  corresponding to the one as in Prop. 3.30.*

*Proof.* The fact that the supermanifold  $\text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}$  is an hermitian superspace is an immediate consequence of Prop. 3.3, while the fact that  $J$  induces an hermitian structure on  $\text{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}$  is a consequence of the comments after Def. 3.2. To see these two structures are the same we need to identify  $\text{osp}(m|2n)/\mathfrak{k} + \mathfrak{p}^+ \cong \mathfrak{p}^-$  with  $\mathfrak{g}_{\mathcal{D}}/\mathfrak{k}_{\mathcal{D}} \cong \mathfrak{p}_{\mathcal{D}}$  and prove that  $J$  induces the same complex structure on  $\mathfrak{p}_{\mathcal{D}}$  as the one inherited by the above identification. ■

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